

TATE CYCLES ON SOME QUATERNIONIC SHIMURA VARIETIES MOD p

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ABSTRACT. Let F be a totally real field in which a prime number $p > 2$ is inert. We continue the study of the (generalized) Goren–Oort strata on quaternionic Shimura varieties over finite extensions of \mathbb{F}_p . We prove that, when the dimension of the quaternionic Shimura variety is even, the Tate conjecture for the special fiber of the quaternionic Shimura variety holds for the cuspidal π -isotypical component, as long as the two unramified Satake parameters at p are not differed by a root of unity.

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1. INTRODUCTION

This paper is the third in a series, in which we study the Goren–Oort stratification of quaternionic Shimura varieties associated to a totally real field F in characteristic p and its applications. In the first paper [TX13⁺a], we gave a global description of each stratum, in terms of a \mathbb{P}^1 -product bundle over another quaternionic Shimura variety. In [TX13⁺b], we apply this result to prove the classicality of overconvergent Hilbert modular forms of small slopes. The aim of this paper is to investigate the Goren–Oort strata by viewing them as special cycles on the special fiber of quaternionic Shimura varieties. This leads to a construction of algebraic cycles of middle codimension on the special fiber of quaternionic Shimura varieties, which, under genericity conditions, coincide with the prediction by the Tate conjecture over finite fields. We start by explaining the underlying philosophy by some examples.

1.1. Hilbert modular surface. Let F be a real quadratic field and $p > 2$ be a prime number that is inert in F/\mathbb{Q} . Let \mathbb{A}_F^∞ be the ring of finite adeles of F , and $K \subset \mathrm{GL}_2(\mathbb{A}_F^\infty)$ be an open compact subgroup hyperspecial at p . We consider the Hilbert modular variety \mathcal{X} of level K ; it admits a smooth integral model \mathcal{X} over $\mathbb{Z}_{(p)}$, with X as its special fiber. Fix a prime number $\ell \neq p$. The main result of Brylinski and Labesse [BL84] says that, up to Frobenius semisimplification, the cuspidal part of the ℓ -adic étale cohomology of X is given as follows

$$H_{\mathrm{et}}^2(X_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)_{\mathrm{cusp}} \cong \bigoplus_{\pi} (\pi^\infty)^K \otimes \rho_\pi|_{\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})}^{\otimes 2},$$

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where the direct sum is taken over all cuspidal automorphic representations whose archimedean components are discrete series of parallel weight 2 and whose p -component is unramified, and ρ_π is the Galois representation associated to π . In particular, the representation ρ_π is unramified at p ; so the tensorial induction is just simply a self tensor product.

We fix an automorphic cuspidal representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ as above. We write $H_{\mathrm{et}}^2(X_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)_\pi := (\pi^\infty)^K \otimes \rho_\pi|_{\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})}^{\otimes 2}$ for the π -isotypical component. Let $\mathrm{Frob}_{p^2} \in \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})$ denote the geometric Frobenius element. We assume that

- the two eigenvalues α and β of $\rho_\pi(\mathrm{Frob}_{p^2})$ are distinct,
- $(\pi^\infty)^K$ is one-dimensional so that $H_{\mathrm{et}}^2(X_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)_\pi \cong \rho_\pi|_{\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^2})}^{\otimes 2}$, and
- the central character of π is trivial.

Here, the first condition is an essential hypothesis, whereas the last two conditions are made to simplify the discussion. The action of Frob_{p^2} on $H_{\mathrm{et}}^2(X_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)_\pi$ has (generalized) eigenvalues α^2, β^2 , and $\alpha\beta = p^2$ which has multiplicity two. In other words, $H_{\mathrm{et}}^2(X_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(1))_\pi$ has a two-dimensional subspace on which the p^2 -Frobenius acts trivially (or more rigorously speaking, unipotently). According to the prediction of the famous Tate Conjecture, this subspace should be generated by cycle classes of X defined over \mathbb{F}_{p^2} .

The main theorem of this paper shows that the cycle classes of Goren–Oort strata ([GO00]) of X span this two dimensional subspace of $H_{\mathrm{et}}^2(X_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(1))_\pi$, and hence we verify the Tate Conjecture in this setting by explicitly exhibiting cycles in X . We explain this with some details now. Recall that the main result of the first paper in this series ([TX13⁺a, Theorem 1.5.1]) implies that there are two collections X_1, X_2 of \mathbb{P}^1 's on X parameterized by the discrete Shimura variety $\mathrm{Sh}_{\infty_1, \infty_2}$ associated to the quaternion algebra B_{∞_1, ∞_2} over F , which ramifies exactly at the two archimedean places of F , of the same level K .¹ From this, we get a natural homomorphism

$$(1.1.1) \quad \bigoplus_{i=1}^2 H_{\mathrm{et}}^0(\mathrm{Sh}_{\infty_1, \infty_2, \overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell) \xrightarrow{\cong} \bigoplus_{i=1}^2 H_{\mathrm{et}}^0(X_{i, \overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell) \xrightarrow{\mathrm{Gysin}} H_{\mathrm{et}}^2(X_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(1))^{\mathrm{Frob}_{p^2}=1},$$

where the second morphism is the Gysin map associated to the closed immersion $X_{i, \overline{\mathbb{F}}_p} \hookrightarrow X_{\overline{\mathbb{F}}_p}$ defined in (4.1.1).² All homomorphisms are equivariant for the prime-to- p Hecke actions. By Jacquet-Langlands correspondence, $(\pi^\infty)^K$ appears on each term of the left hand side of (1.1.1) with multiplicity one. Taking the π -isotypical parts (or more precisely the $(\pi^\infty)^K$ -isotypical parts) of (1.1.1) gives rise to a homomorphism

$$\bigoplus_{i=1}^2 \overline{\mathbb{Q}}_\ell \xrightarrow{\mathrm{Gysin}} H_{\mathrm{et}}^2(X_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(1))_\pi^{\mathrm{Frob}_{p^2}=1}.$$

Our main result of this paper says that this is an isomorphism. To show this, we consider the natural restriction map:

$$H_{\mathrm{et}}^2(X_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(1)) \xrightarrow{\mathrm{res}} \bigoplus_{i=1}^2 H_{\mathrm{et}}^2(X_{i, \overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(1)) \cong \bigoplus_{i=1}^2 H_{\mathrm{et}}^0(\mathrm{Sh}_{\infty_1, \infty_2, \overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell).$$

¹As pointed out by the referee, the existence of two families of \mathbb{P}^1 on Hilbert modular surfaces was first observed by Pappas in his Ph.D. thesis.

²If we pretend that X is proper, then the Gysin map is dual to the restriction map $H_{\mathrm{et}}^2(X_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell) \rightarrow H_{\mathrm{et}}^2(X_{i, \overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)$ under the Poincaré dualities on X and X_i .

Its composition with (1.1.1) gives an endomorphism of $\bigoplus_{i=1}^2 H^0(\mathrm{Sh}_{\infty_1, \infty_2, \overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)$. The key point is to show that the projection to the π -isotypical component of this endomorphism is given by

$$\begin{pmatrix} -2p & \alpha + \beta \\ \alpha + \beta & -2p \end{pmatrix},$$

whose determinant is equal to $-(\alpha - \beta)^2$. Here, the entries $\alpha + \beta$ come from the fact that the morphisms mapping the intersection $X_1 \cap X_2$ to $\mathrm{Sh}_{\infty_1, \infty_2}$ using the two \mathbb{P}^1 -parametrizations exactly give the Hecke correspondence T_p of $\mathrm{Sh}_{\infty_1, \infty_2}$,³ and hence we see the evaluation of T_p at π , which is $\alpha + \beta$. This matrix should be viewed as the π -projection of the intersection matrix of the Goren–Oort strata X_1 and X_2 . From this, we see that when $\alpha \neq \beta$, these Goren–Oort strata give rise to all Tate cycles in the π -isotypical component of the special fiber of the Hilbert modular surface.

In contrast, if $\alpha = \beta$, $H_{\mathrm{et}}^2(X_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(1))_\pi$ is expected to have four dimensional Tate classes. However, the image of the classes of Goren–Oort strata in $H_{\mathrm{et}}^2(X_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(1))_\pi$ probably only contributes to a one-dimensional subspace (see Example 4.6 for the discussion).

Kartik Prasanna suggested to us that one might be able to obtain finer information when $\alpha = \beta \in \{p, -p\}$. In this case, if we consider the action of Frob_p on $H_{\mathrm{et}}^2(X_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(1))_\pi$ (as opposed to the Frob_{p^2} -action), the eigenvalues are α with multiplicity three and $-\alpha$ with multiplicity one. One can show that the image of the Goren–Oort strata contains (and is expected to be equal to) the $(-\alpha)$ -eigenspace. It is the α -eigenspace that is “larger than usual”, which causes the cycle map to be zero.⁴ See also Remark 4.6.

1.2. Hilbert modular four-folds. As one tries to generalize the result above to higher dimensional cases, one finds quickly that the Goren–Oort stratification does not provide enough interesting cycles. Take a Hilbert modular four-fold \mathcal{X} as an example, where the totally real field F defining it has degree 4 and p is inert in F/\mathbb{Q} , and the level structure $K \subseteq \mathrm{GL}_2(\mathbb{A}_F^\infty)$ is hyperspecial at p . Hence \mathcal{X} has an integral model over $\mathbb{Z}_{(p)}$, and let X denote its special fiber. Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ with trivial central character whose archimedean components are discrete series of parallel weight 2 and whose p -adic component is unramified. We assume again that $(\pi^\infty)^K$ is one-dimensional. Then the π -isotypical component $H_{\mathrm{et}}^4(X_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell)_\pi$ is isomorphic to $\rho_\pi|_{\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^4})}^{\otimes 4}$ up to Frobenius semisimplification. Let α and β be the (generalized) eigenvalues of $\rho_\pi(\mathrm{Frob}_{p^4})$ so that $\alpha\beta = p^4$. The essential hypothesis in this case is that $\alpha/\beta \neq \pm 1$. The same computation as before shows that the multiplicity of Frob_{p^4} -eigenvalue 1 in $H_{\mathrm{et}}^4(X_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(2))_\pi$ is $\binom{4}{2} = 6$. As a result, we should expect 6 collections of certain (2-dimensional) strata of X which contribute to this 6-dimensional subspace.

We recall the following description of the Goren–Oort strata in [TX13⁺a, §1.5.3]. There are indeed 6 collections of two-dimensional Goren–Oort strata X_{ij} for $\{i, j\} \subset \{0, \dots, 3\}$. Unfortunately, only two of them X_{02} and X_{13} contribute to the correct Tate classes; they are $(\mathbb{P}^1)^2$ -bundles parametrized by the discrete Shimura varieties for the quaternion algebra $B_{\infty_0, \dots, \infty_3}$ over F (which ramifies exactly at all archimedean places). Other strata are \mathbb{P}^1 -bundles over certain Shimura curves, and they do not contribute to any Tate cycles in $H_{\mathrm{et}}^4(X_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(2))_\pi$, since Shimura curves do not have interesting H^0 's.

³In general, we arrive at certain twist of the Hecke correspondence T_p . But the twist disappears under the assumption on the trivialness of the central character.

⁴One should compare this for example with the Gross–Zagier formula in the case when the elliptic curve has rank ≥ 2 . In that case, the Mordell–Weil rank is “larger than usual”. So the Heegner point is “unwilling to pick out a canonical rank one subgroup of the Mordell–Weil group”, and hence has to be torsion. For the same philosophy, in our case, the cycle class from the Goren–Oort strata is expected to be zero in the “unusually large” $(-\alpha)$ -eigenspace.

To get enough strata that contribute to the Tate cycles of $H_{\text{et}}^4(X_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(2))_\pi$, we need to look at all codimension one strata X_0, \dots, X_3 first. Each X_i is a \mathbb{P}^1 -bundle over the Shimura surface for the quaternion algebra $B_{\infty_i, \infty_{i-1}}$. If we consider the Goren–Oort stratification of the Shimura surface for $B_{\infty_i, \infty_{i-1}}$ and take the corresponding \mathbb{P}^1 -bundle, we will get two 2-dimensional subvarieties $X_{i,1}$ and $X_{i,2}$ of X_i . In fact one of them is equal to a Goren–Oort stratum (either X_{02} or X_{13}) and the other one is a completely new collection of subvarieties of X , which is a family of \mathbb{P}^1 -bundles over \mathbb{P}^1 parametrized by the discrete Shimura variety for $B_{\infty_0, \dots, \infty_3}$. We call them (*generalized*) *Goren–Oort cycles*. In fact, each irreducible component is isomorphic to $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(p) \oplus \mathcal{O}_{\mathbb{P}^1}(-p))$ (see Example 3.9). One can characterize these new subvarieties by looking at the p^2 -torsion of the universal abelian variety. To sum up, we obtain 4 new two-dimensional subvarieties. Our main theorem says that they together with the two Goren–Oort strata, span the 6-dimensional Tate classes in $H_{\text{et}}^4(X_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(1))_\pi$.⁵

More generally, we prove the following.

Theorem 1.3 (corollary of Theorem 4.4). *Let F be a totally real field of even degree $g = [F : \mathbb{Q}]$, and let p be a rational prime inert in F . Let X denote the special fiber over \mathbb{F}_{p^g} of the Hilbert modular variety of some level $K \subseteq \text{GL}_2(\mathbb{A}_F^\infty)$ hyperspecial at p . Fix an automorphic representation π of $\text{GL}_2(\mathbb{A}_F)$ whose archimedean components are all discrete series of weight 2 (the lowest weight) with trivial central character. Suppose that π_p is an unramified principal series whose two Satake parameters do not differ by an n -th root of unity for any $n \leq g/2$.*

Assume that $(\pi^\infty)^K$ is one-dimensional. Then Tate Conjecture holds for the π -isotypical component $H_{\text{et}}^g(X_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(\frac{g}{2}))_\pi$. More precisely, the generalized invariant subspace of $H_{\text{et}}^g(X_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(\frac{g}{2}))_\pi$ under the action of $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^g})$ has dimension $\binom{g}{2}$, and it is generated by the cycle classes of Goren–Oort cycles on X .

The condition on the Satake parameters is equivalent to saying that all Frob_{p^g} -eigenvalues on $\rho_\pi^{\otimes g}$ are distinct. We emphasize once again that this condition is essential. The contribution from the (generalized) Goren–Oort cycles can be degenerate if the two parameters are the same. See Remark 4.6.

One might wonder whether this result adds new knowledge on the semi-simplicity of the Frobenius action on the cohomology groups of Hilbert modular varieties. Unfortunately, our theorem applies only when the two Frobenius eigenvalues α and β are distinct. In this case, $\rho_\pi(\text{Frob}_{p^g})$ is already semisimple. If the tensorial induction of ρ_π is an irreducible representation of $\text{Gal}_{\mathbb{Q}}$ (as opposed to $\text{Gal}_{\mathbb{F}_{p^g}}$), then the semisimplicity of Frob_{p^g} on the π -isotypical component is already known. Note also that Nekovář [Ne15⁺] proved recently the semi-simplicity of Galois representation appearing in $H_{\text{et}}^g(\mathcal{X}_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)$, using Eichler–Shimura relations.

1.4. Framework for Goren–Oort cycles. The explicit verification of Tate Conjecture is merely an “extremal” case of the following study of Goren–Oort cycles. We still start with the Galois side. Let F be a totally real field of degree g over \mathbb{Q} and let p be a prime inert in F/\mathbb{Q} . We fix a natural number $r \leq g/2$. We consider a regular multiweight (\underline{k}, w) , that is a collection of integers, where $\underline{k} = (k_1, \dots, k_g)$ with $k_i \geq 2$ and $k_i \equiv w \pmod{2}$.

As before, we consider the special fiber X over \mathbb{F}_p of the Hilbert modular variety, taking the limit over all prime-to- p level structures, but fixing the open compact subgroup at p to be hyperspecial. There is an automorphic local system $\mathcal{L}^{(\underline{k}, w)}$ over X . We fix a cuspidal automorphic representation π of $\text{GL}_2(\mathbb{A}_F)$ associated to holomorphic Hilbert modular forms of weight (\underline{k}, w) . We focus on the π -isotypical component

$$H_{\text{et}}^g(X_{\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k}, w)})[\pi] := \text{Hom}_{\text{GL}_2(\mathbb{A}_F^{\infty, p})}(\pi^{\infty, p}, H_{\text{et}}^g(X_{\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k}, w)})),$$

⁵We recently learned that the existence of these 6 collections of two-dimensional subvarieties of X (which form the supersingular locus of X) were previously known by Chia-Fu Yu [Yu03].

which is (up to Frobenius semisimplification) isomorphic to $\rho_\pi|_{\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_{p^g})}^{\otimes g}$, where ρ_π is the Galois representation associated to π . Let α and β denote the two eigenvalues of $\rho_\pi(\text{Frob}_{p^g})$. We assume that α/β is *not* a $2n$ th root of unity for $n \leq g$.⁶

We consider the p^g -Frobenius eigenvalue $\alpha^{g-r}\beta^r$ for some positive integer $r \leq \frac{g}{2}$. The associated generalized eigenspace has dimension $\binom{g}{r}$. The case $r = \frac{g}{2}$ gives rise to Tate cycles discussed above. Our main result of this paper is the following.

Theorem 1.5 (Theorem 4.4). *Suppose that α/β is not a $2n$ -th root of unity for $n \leq g$. There exist $\binom{g}{r}$ subvarieties $X_1, \dots, X_{\binom{g}{r}}$ (explicitly defined later) of $X_{\mathbb{F}_{p^g}}$ with the following properties:*

- (1) *Each X_i is an r -times iterated \mathbb{P}^1 -bundle over the special fiber of a Shimura variety⁷ associated with a quaternion algebra over F which ramifies exactly at $2r$ archimedean places.*
- (2) *The direct sum of the Gysin maps*

$$(1.5.1) \quad \bigoplus_{i=1}^{\binom{g}{r}} H_{\text{et}}^{g-2r}(X_{i, \overline{\mathbb{F}}_p}, \mathcal{L}^{(k, w)}|_{X_i})[\pi] \longrightarrow H_{\text{et}}^g(X_{\overline{\mathbb{F}}_p}, \mathcal{L}^{(k, w)}(r))[\pi]$$

is an isomorphism on the generalized eigenspace for $\text{Frob}_{p^g} = \alpha^{g-r}\beta^r/p^{gr}$.

Theorem 1.3 is a special (and degenerate) case of this theorem (except for a less restricted condition on α/β). We in fact prove in Theorem 4.4 a stronger result than stated here for general quaternionic Shimura varieties.

Remark 1.6. The name Goren–Oort cycles is slightly misleading. In fact, when $r = g/2$, one can prove that the union of the X_i ’s considered in Theorem 1.5 is the supersingular locus of X . More generally, as Xinwen Zhu pointed out to us, the union of the X_i ’s considered in Theorem 1.5 is the closure of the *Newton stratum* of the Hilbert modular variety X , with slopes $(\frac{r}{g}, \dots, \frac{r}{g}, \frac{g-r}{g}, \dots, \frac{g-r}{g})$.

Remark 1.7. In the view of the previous remark, we point out the key philosophy suggested by Theorem 1.5: *the irreducible components of the Newton strata of the Hilbert modular varieties (generically) contribute to certain Frobenius eigenspace of the cohomology of the Hilbert modular varieties. In particular, the supersingular locus contributes to the Tate classes in the cohomology. Moreover, the (generic) dimension of the Frobenius eigenspace determines the number of irreducible components in the corresponding Newton stratum.*

We expect this philosophy to continue to hold for more general Shimura varieties, as elaborated in the later joint work of the authors and David Helm [HTX14⁺], as well as the forthcoming joint work of Xinwen Zhu and the second author [XZ⁺].

1.8. Intersection matrix. Before giving the construction of the subvarieties X_i , let us first explain the essential idea in the proof of Theorem 1.5(2). A straightforward computation shows that the generalized eigenspace for $\text{Frob}_{p^g} = \alpha^{g-r}\beta^r/p^{gr}$ on both sides of (1.5.1) have the same dimension. Hence, to prove our Theorem, it is enough to show that the left vertical homomorphism in the following diagram is an isomorphism when restricted to the generalized eigenspace for $\text{Frob}_{p^g} =$

⁶The reason that we have $2n$ th (as opposed to n th) root of unity here is purely technical. See Remark 4.5(3).

⁷In fact, we need a slightly funny choice of Deligne homomorphism for these quaternionic Shimura varieties. We refer to the main context of the paper for details.



$$\alpha^{g-r} \beta^r / p^{gr}.$$

$$\begin{array}{ccc} \bigoplus_{i=1}^{\binom{g}{r}} H_{\text{et}}^{g-2r}(X_{i, \overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k}, w)}|_{X_i})[\pi] & \xrightarrow[\text{Gysin}]{(1.5.1)} & H_{\text{et}}^g(X_{\overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k}, w)}(r))[\pi] \\ \downarrow \Psi & & \downarrow \text{restriction} \\ \bigoplus_{i=1}^{\binom{g}{r}} H_{\text{et}}^{g-2r}(X_{i, \overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k}, w)}|_{X_i})[\pi] & \xrightarrow[\cong]{\cup(c_1(\mathbb{P}^1))^r} & \bigoplus_{i=1}^{\binom{g}{r}} H_{\text{et}}^g(X_{i, \overline{\mathbb{F}}_p}, \mathcal{L}^{(\underline{k}, w)}|_{X_i}(r))[\pi], \end{array}$$

In other words, we are reduced to showing that a certain $\binom{g}{r} \times \binom{g}{r}$ -matrix is invertible. When $g = 2r$ and (\underline{k}, w) is of parallel weight 2, this is the intersection matrix of X_i 's. When both g and r are relatively small, it is not extremely difficult to compute this matrix and its determinant. However, it appears to be a non-trivial problem to prove the invertibility of such a matrix for general g and r .

In this paper, we will show that the matrix can be computed in a completely combinatorial way, and it is related to a version of Gram determinant for periodic semi-meanders, which has been well studied by mathematical physicists [MS13, GL98]. We confess that this potential link with mathematical physics is probably a coincidence. It only suggests a strong relation to representation theory, to which the mathematics physics problem is also related. See Remark 3.4.

1.9. Goren–Oort cycles: construction. The best way (so far) to parametrize the Goren–Oort cycles is to use periodic semi-meanders (mostly for the benefit of later computation of the Gysin-restriction matrix). As before, we take F to be a totally real field of degree g in which p is inert.

A *periodic semi-meander* of g nodes is a graph where g nodes are aligned equidistant on a section of a vertical cylinder, and are either connected pairwise by non-intersecting curves (called *arcs*) drawn above the section, or connected by a straight line (called *semi-lines*) to $+\infty$ on the top of the cylinder. We use r to denote the number of arcs. For example,  and  are both semi-meanders of 6 points with $r = 2$ and 3 respectively. An elementary computation shows that there are $\binom{g}{r}$ semi-meanders of g nodes with r arcs ($r \leq \frac{g}{2}$).

We label the set of p -adic embeddings $\mathcal{O}_F \hookrightarrow W(\overline{\mathbb{F}}_p) = \widehat{\mathbb{Z}}_p^{\text{ur}}$ by τ_1, \dots, τ_g so that $\sigma \circ \tau_i = \tau_{i+1}$, where σ is the absolute Frobenius $W(\overline{\mathbb{F}}_p) \rightarrow W(\overline{\mathbb{F}}_p)$ and $\tau_i = \tau_{i \bmod g}$. Let X denote the special fiber of the Hilbert modular variety. For each $\overline{\mathbb{F}}_p$ -point $x \in X$, we denote by A_x the fiber at x of the universal abelian scheme. Let \mathcal{D}_x denote the covariant Dieudonné module of the p -divisible group of A_x . It is a free $W(\overline{\mathbb{F}}_p)$ -module of rank $2g$. The \mathcal{O}_F -action on A_x induces a natural direct sum decomposition $\mathcal{D}_x \cong \bigoplus_{i=1}^g \mathcal{D}_{x,i}$, where $\mathcal{D}_{x,i}$ is the direct summand of \mathcal{D}_x on which \mathcal{O}_F acts through τ_i . Each $\mathcal{D}_{x,i}$ is free of rank two over $W(\overline{\mathbb{F}}_p)$.


The Verschiebung induces a σ^{-1} -semilinear map $V_i : \mathcal{D}_{x,i+1} \rightarrow \mathcal{D}_{x,i}$. The image $V_i(\mathcal{D}_{x,i+1})/p\mathcal{D}_{x,i}$ is canonically isomorphic to $\omega_{A_x^\vee, i}$, the τ_i -component of the invariant 1-differentials on A_x^\vee . The latter is a one-dimensional $\overline{\mathbb{F}}_p$ -vector space.

We fix $r \leq g/2$. To each semi-meander \mathbf{a} considered above, we associate a subvariety $X_{\mathbf{a}}$ of X whose $\overline{\mathbb{F}}_p$ -points x are characterized as follows:

- For each curve connecting a and $a + d$ for some odd number $d < g$, we require that

$$V_a \circ V_{a+1} \circ \dots \circ V_{a+d}(\mathcal{D}_{x, a+d+1}) \subseteq p^{(d+1)/2} \mathcal{D}_{x, a} \text{ or equivalently } p^{(d+1)/2} | V_a \circ V_{a+1} \circ \dots \circ V_{a+d}.$$

In fact, the inclusion forces an equality.

For example, the condition for the semi-meander  (with $g = 7$) is

$$p \mid V_0 V_1, \quad p \mid V_2 V_3, \quad \text{and } p \mid V_6 \frac{V_0 V_1}{p} \frac{V_2 V_3}{p} V_4.$$

Iteratively applying (the divisor case of) the main theorem of [TX13⁺a] (see §1.5 of *loc. cit.*), it is not difficult to see that X_a is an r -times iterated \mathbb{P}^1 -bundle over the special fiber of quaternionic Shimura variety⁸ for the quaternion algebra which ramifies at all archimedean places corresponding to those nodes which are linked to arcs.

The main computation of this paper is to show that the Gysin-restriction matrix is roughly given by the natural bilinear form on the vector space spanned by the periodic semi-meanders. See Subsection 3.2 for the definition and Theorem 4.3 for the precise statement.

Finally, we make a technical remark: our results involve transferring constructions from the unitary setup to the quaternion case. This is the origin of most notational complications. Moreover, certain descriptions of the Goren–Oort cycles have ambiguity, but the ambiguity does not affect the proof of the main theorem.

Structure of the paper. In Section 2, we recall necessary facts about Goren–Oort stratification from [TX13⁺a]. Some of the proofs are mostly book-keeping but technical. The readers may skip them for the first time reading. In Section 3, we first recall the combinatorics about semi-meanders and then give the definition of the Goren–Oort cycles associated to each periodic semi-meander. In Section 4, we state our main Theorem 4.4 and prove them modulo Theorem 4.3, which says that the Gysin-restriction matrix for Goren–Oort cycles is roughly the same as the Gram matrix of the corresponding periodic semi-meanders. This key theorem is proved in Section 5. The appendix includes a proof of the description of the cohomology of quaternionic Shimura varieties. This is well known to the experts but we include it there for completeness.

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2. GOREN–OORT STRATIFICATION

We first recall the Goren–Oort stratification of the special fiber of quaternionic Shimura varieties and their descriptions, following [TX13⁺a]. We tailor our discussion to later application and hence we will focus on certain special cases discussed in *loc. cit.*

⁸More rigorously, we have a funny choice of Deligne homomorphism for this quaternionic Shimura variety.

2.1. Notation. For a field F , we use Gal_F to denote its absolute Galois group. For a number field F , we write \mathbb{A}_F (resp. $\mathbb{A}_F^\infty, \mathbb{A}_F^{\infty,p}$) for its ring of adeles (resp. finite adeles, finite adeles away from a rational prime p). When $F = \mathbb{Q}$, we suppress the subscript F , e.g. by writing \mathbb{A}^∞ . Let \underline{p}_F denote the idele of \mathbb{A}_F^∞ which is p at all p -adic places and trivial elsewhere. We also normalize the Artin reciprocity map $\text{Art} : \mathbb{A}_F^\times / F^\times \rightarrow \text{Gal}_F^{\text{ab}}$ so that a local uniformizer at a finite place v corresponds to a *geometric* Frobenius element at v .

We fix F a totally real field of degree $g > 1$ over \mathbb{Q} . Let Σ denote the set of places of F , and Σ_∞ the subset of all real places. We fix a prime number $p > 2$ *inert* in the extension F/\mathbb{Q} .⁹ We put $\mathfrak{p} = p\mathcal{O}_F$, $F_{\mathfrak{p}}$ the completion of F at \mathfrak{p} , $\mathcal{O}_{\mathfrak{p}}$ the valuation ring, and $k_{\mathfrak{p}}$ the residue field.

We fix an isomorphism $\iota_p : \mathbb{C} \simeq \overline{\mathbb{Q}}_p$. Let \mathbb{Q}_{p^g} denote the unramified extension of \mathbb{Q}_p of degree g in $\overline{\mathbb{Q}}_p$; let \mathbb{Z}_{p^g} be its valuation ring. Post-composition with ι_p identifies $\Sigma_\infty = \text{Hom}(F, \mathbb{R}) \cong \text{Hom}(F, \mathbb{Q}_{p^g}) \cong \text{Hom}(\mathcal{O}_F, \mathbb{F}_{p^g})$. In particular, the absolute Frobenius σ acts on Σ_∞ by sending $\tau \in \Sigma_\infty$ to $\sigma\tau := \sigma \circ \tau$; this makes Σ_∞ into one cycle. Let \mathbb{Q}_p^{ur} denote the maximal unramified extension of \mathbb{Q}_p , and \mathbb{Z}_p^{ur} denote its valuation ring.

For a finite field \mathbb{F}_q , we denote by $\text{Frob}_q \in \text{Gal}_{\mathbb{F}_q}$ the *geometric* Frobenius element.

2.2. Quaternionic Shimura varieties. Let \mathbf{S} be a set of places of F of even cardinality such that $\mathfrak{p} \notin \mathbf{S}$. Put $\mathbf{S}_\infty = \mathbf{S} \cap \Sigma_\infty$ and $\mathbf{S}_\infty^c = \Sigma_\infty - \mathbf{S}_\infty$,¹⁰ and $d = \#\mathbf{S}_\infty^c$. We also fix a subset \mathbf{T} of \mathbf{S}_∞ . We denote by $B_{\mathbf{S}}$ the quaternion algebra over F ramified exactly at \mathbf{S} . Let $G_{\mathbf{S},\mathbf{T}} = \text{Res}_{F/\mathbb{Q}}(B_{\mathbf{S}}^\times)$ be the associated \mathbb{Q} -algebraic group. Here we inserted the subscript \mathbf{T} because we use the following *Deligne homomorphism*

$$h_{\mathbf{S},\mathbf{T}} : \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times \longrightarrow G_{\mathbf{S},\mathbf{T}}(\mathbb{R}) \cong (\mathbb{H}^\times)^{\mathbf{S}_\infty - \mathbf{T}} \times (\mathbb{H}^\times)^{\mathbf{T}} \times \text{GL}_2(\mathbb{R})^{\mathbf{S}_\infty^c}$$

$$x + y\mathbf{i} \longmapsto \left((1, \dots, 1), (x^2 + y^2, \dots, x^2 + y^2), \left(\begin{pmatrix} x & y \\ -y & x \end{pmatrix}, \dots, \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \right) \right).$$

When $\mathbf{T} = \emptyset$, the Deligne homomorphism $h_{\mathbf{S},\emptyset}$ is the same as $h_{\mathbf{S}}$ considered in [TX13⁺a, §3.1]. The $G_{\mathbf{S},\mathbf{T}}(\mathbb{R})$ -conjugacy class of $h_{\mathbf{S},\mathbf{T}}$ is independent of \mathbf{T} and is isomorphic to $\mathfrak{H}_{\mathbf{S}} := (\mathfrak{h}^\pm)^{\mathbf{S}_\infty^c}$, where $\mathfrak{h}^\pm = \mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$. Consider the Hodge cocharacter

$$\mu_{\mathbf{S},\mathbf{T}} : \mathbb{G}_{m,\mathbb{C}} \xrightarrow{z \mapsto (z,1)} \mathbb{S}_{\mathbb{C}} \cong \mathbb{G}_{m,\mathbb{C}} \times \mathbb{G}_{m,\mathbb{C}} \xrightarrow{h_{\mathbf{S},\mathbf{T}}} G_{\mathbf{S},\mathbf{T},\mathbb{C}}.$$

Here, the composition of the natural inclusion $\mathbb{C}^\times = \mathbb{S}(\mathbb{R}) \hookrightarrow \mathbb{S}(\mathbb{C})$ with the first (resp. second) projection $\mathbb{S}(\mathbb{C}) \rightarrow \mathbb{C}^\times$ is the identity map (resp. the complex conjugation).

The reflex field $F_{\mathbf{S},\mathbf{T}}$, i.e. the field of definition of the conjugacy class of $\mu_{\mathbf{S},\mathbf{T}}$, is a finite extension of \mathbb{Q} sitting inside \mathbb{C} and hence inside $\overline{\mathbb{Q}}_p$ via ι_p . It is clear that the p -adic closure of $F_{\mathbf{S},\mathbf{T}}$ in $\overline{\mathbb{Q}}_p$ is contained in \mathbb{Q}_{p^g} , the unramified extension of \mathbb{Q}_p of degree g in $\overline{\mathbb{Q}}_p$. Instead of working with occasional smaller reflex field, we are content with working with Shimura varieties over \mathbb{Q}_{p^g} .

We fix an isomorphism $G_{\mathbf{S},\mathbf{T}}(\mathbb{Q}_p) \simeq \text{GL}_2(F_{\mathfrak{p}})$ and put $K_p = \text{GL}_2(\mathcal{O}_{\mathfrak{p}})$ or occasionally $\text{Iw}_p := \begin{pmatrix} \mathcal{O}_{\mathfrak{p}}^\times & \mathcal{O}_{\mathfrak{p}} \\ p\mathcal{O}_{\mathfrak{p}} & \mathcal{O}_{\mathfrak{p}}^\times \end{pmatrix}$ when $\mathbf{S}_\infty^c = \emptyset$. We will only consider open compact subgroups $K \subseteq G_{\mathbf{S},\mathbf{T}}(\mathbb{A}^\infty)$ of the form $K = K_p K^p$ with K^p an open compact subgroup of $G_{\mathbf{S},\mathbf{T}}(\mathbb{A}^{\infty,p})$.¹¹ For such a K , we have a Shimura

⁹Although most of our argument works equally well when p is only assumed to be unramified, we insist to assume that p is inert which largely simplifies the notation so that the proof of the main result is more accessible. But see Remark 4.5(1).

¹⁰Note that the upper script c was used to denote complex conjugation in [TX13⁺a]. In this paper, we however use it to mean taking the set theoretic complement.

¹¹In earlier papers of this series, the open compact subgroup K was denoted by $K_{\mathbf{S}}$. We choose to drop the subscript because for all \mathbf{S} we encounter later, the group $G_{\mathbf{S}}(\mathbb{A}^\infty)$ are isomorphic, and hence we can naturally identify the $K_{\mathbf{S}}$'s for different \mathbf{S} 's.

variety $\mathcal{Sh}_K(G_{\mathbf{S},\mathbf{T}})$ defined over \mathbb{Q}_{p^g} , whose \mathbb{C} -points (via ι_p) are given by

$$\mathcal{Sh}_K(G_{\mathbf{S},\mathbf{T}})(\mathbb{C}) = G_{\mathbf{S},\mathbf{T}}(\mathbb{Q}) \backslash \mathfrak{H}_{\mathbf{S}} \times G_{\mathbf{S},\mathbf{T}}(\mathbb{A}^\infty) / K.$$

We put $\mathcal{Sh}_{K_p}(G_{\mathbf{S},\mathbf{T}}) := \varprojlim_{K_p} \mathcal{Sh}_{K_p K_p}(G_{\mathbf{S},\mathbf{T}})$. This Shimura variety has dimension $d = \#\mathbf{S}_\infty^c$. There is a natural morphism of geometric connected components

$$(2.2.1) \quad \pi_0(\mathcal{Sh}_{K_p}(G_{\mathbf{S},\mathbf{T}})_{\overline{\mathbb{Q}}_p}) \longrightarrow F_+^{\times, \text{cl}} \backslash \mathbb{A}_F^{\infty, \times} / \mathcal{O}_{\mathfrak{p}}^\times,$$

where F_+^\times is the subgroup of totally positive elements of F^\times , and the superscript cl stands for taking closure in the corresponding topological space. The morphism (2.2.1) is an isomorphism if $\mathbf{S}_\infty^c \neq \emptyset$ by [De71, Théorème 2.4]. Following the convention in [TX13⁺a, §2.11], *we shall call the preimage of an element $\mathbf{x} \in F_+^{\times, \text{cl}} \backslash \mathbb{A}_F^{\infty, \times} / \mathcal{O}_{\mathfrak{p}}^\times$ under the map (2.2.1) a geometric connected component, although it is not geometrically connected when $\mathbf{S}_\infty^c = \emptyset$. The preimage of $\mathbf{1}$ is called the neutral geometric connected component, which we denote by $\mathcal{Sh}_{K_p}(G_{\mathbf{S},\mathbf{T}})_{\overline{\mathbb{Q}}_p}^\circ$.*

Note that, for different choices of \mathbf{T} , the Shimura varieties $\mathcal{Sh}_K(G_{\mathbf{S},\mathbf{T}})$ are isomorphic over $\overline{\mathbb{Q}}_p$ (in fact over $\overline{\mathbb{Q}}$ if we have not p -adically completed the reflex field), but the actions of $\text{Gal}(\overline{\mathbb{Q}}_p / \mathbb{Q}_{p^g})$ depend on \mathbf{T} . By Shimura reciprocity law (cf. [De71] or [TX13⁺a, §2.7]), the action of $\text{Gal}(\overline{\mathbb{Q}}_p / \mathbb{Q}_{p^g})$ on $\pi_0(\mathcal{Sh}_{K_p}(G_{\mathbf{S},\mathbf{T}})_{\overline{\mathbb{Q}}_p})$ factors through $\text{Gal}_{\mathbb{F}_{p^g}} \cong \text{Gal}(\mathbb{Z}_p^{\text{ur}} / \mathbb{Z}_{p^g})$, so that the connected components of $\mathcal{Sh}_{K_p}(G_{\mathbf{S},\mathbf{T}})_{\overline{\mathbb{Q}}_p}$ are actually defined over \mathbb{Q}_p^{ur} , the maximal unramified extension of \mathbb{Q}_p . More precisely, the action of the geometric Frobenius of \mathbb{F}_{p^g} on $F_+^{\times, \text{cl}} \backslash \mathbb{A}_F^{\infty, \times} / \mathcal{O}_{\mathfrak{p}}^\times$, induced through the homomorphism (2.2.1), is given by multiplication by the finite idele

$$(2.2.2) \quad (\underline{p}_F)^{(2\#\mathbf{T} + \#\mathbf{S}_\infty^c)} \in F_+^{\times, \text{cl}} \backslash \mathbb{A}_F^{\infty, \times} / \mathcal{O}_{\mathfrak{p}}^\times.^{12}$$

This determines a reciprocity map:

$$\text{Rec}_p: \text{Gal}_{\mathbb{F}_{p^g}} \longrightarrow F_+^{\times, \text{cl}} \backslash \mathbb{A}_F^{\infty, \times} / \mathcal{O}_{\mathfrak{p}}^\times.$$

Write $\nu: G_{\mathbf{S},\mathbf{T}} \rightarrow \text{Res}_{F/\mathbb{Q}}(\mathbb{G}_m)$ for the reduced norm homomorphism. Following Deligne's recipe [De79] of connected Shimura varieties, we put

$$(2.2.3) \quad \mathcal{G}_{\mathbf{S},\mathbf{T},p} := (G_{\mathbf{S},\mathbf{T}}(\mathbb{A}^{\infty,p}) / \mathcal{O}_{F,(p)}^{\times, \text{cl}}) \times \text{Gal}_{\mathbb{F}_{p^g}}^{13}$$

and define $\mathcal{E}_{G_{\mathbf{S},\mathbf{T}}}$ to be the subgroup of $\mathcal{G}_{\mathbf{S},\mathbf{T},p}$ consisting of pairs (x, σ) such that $\nu(x)$ is equal to $\text{Rec}_p(\sigma)^{-1}$. Here, $\mathcal{O}_{F,(p)}^{\times, \text{cl}}$ denotes the closure of $\mathcal{O}_{F,(p)}^\times$ in $G_{\mathbf{S},\mathbf{T}}(\mathbb{A}^{\infty,p})$.

The limit $\mathcal{Sh}_{K_p}(G_{\mathbf{S},\mathbf{T}})_{\mathbb{Q}_p^{\text{ur}}}$ carries an action by $\mathcal{G}_{\mathbf{S},\mathbf{T},p}$, and $\mathcal{E}_{G_{\mathbf{S},\mathbf{T}}}$ is the stablizer of each geometric connected component. Conversely, if $\mathcal{Sh}_{K_p}(G_{\mathbf{S},\mathbf{T}})_{\mathbb{Q}_p^{\text{ur}}}^\bullet$ is a geometric connected component, one can recover $\mathcal{Sh}_{K_p}(G_{\mathbf{S},\mathbf{T}})$ from $\mathcal{Sh}_{K_p}(G_{\mathbf{S},\mathbf{T}})_{\mathbb{Q}_p^{\text{ur}}}^\bullet$ by first forming the product

$$\mathcal{Sh}_{K_p}(G_{\mathbf{S},\mathbf{T}})_{\mathbb{Q}_p^{\text{ur}}}^\bullet \times_{\mathcal{E}_{G_{\mathbf{S},\mathbf{T}}}} \mathcal{G}_{\mathbf{S},\mathbf{T},p}$$

and then take the Galois descend to \mathbb{Q}_{p^g} .

Notation 2.3. Note that $G_{\mathbf{S},\mathbf{T}}(\mathbb{A}^\infty)$ depends only on the finite places contained in \mathbf{S} . In later applications, we will consider only pairs of subsets $(\mathbf{S}', \mathbf{T}')$ such that \mathbf{S}' contains the same finite places as \mathbf{S} . In that case, we will fix an isomorphism $G_{\mathbf{S}',\mathbf{T}'}(\mathbb{A}^\infty) \cong G_{\mathbf{S},\mathbf{T}}(\mathbb{A}^\infty)$, and denote them uniformly by $G(\mathbb{A}^\infty)$ when no confusions arise. Similarly, we have its subgroup $G(\mathbb{A}^{\infty,p}) \subseteq G(\mathbb{A}^\infty)$

¹²When $\mathbf{S}_\infty^c = \emptyset$ or equivalently when $\mathcal{Sh}_{K_p}(G_{\mathbf{S},\mathbf{T}})$ is a zero-dimensional Shimura variety, the action of Frob_{p^g} is given by multiplication by the finite idele $(\underline{p}_F)^{\#\mathbf{T}}$ in the center $\text{Res}_{F/\mathbb{Q}} \mathbb{G}_m$ of $G_{\mathbf{S},\mathbf{T}}$. This gives the canonical model for the discrete Shimura variety in the sense of [TX13⁺a, 2.8].

¹³Comparing with [TX13⁺a, (2.11.3)], we dropped the star extension because the center of $G_{\mathbf{S},\mathbf{T}}$ is $\text{Res}_{F/\mathbb{Q}} \mathbb{G}_m$, which has trivial first cohomology. We also include the Galois part into the definition of \mathcal{G} to simplify notation here.

consisting of elements whose p -component is trivial. Thus, we may view K (resp. K^p) as an open compact subgroup of $G(\mathbb{A}^\infty)$ (resp. $G(\mathbb{A}^{\infty,p})$).

Under this identification, the group $\mathcal{G}_{\mathbf{S},\mathbf{T},p}$ is independent of \mathbf{S}, \mathbf{T} , and we henceforth write \mathcal{G}_p for it. Its subgroup $\mathcal{E}_{G_{\mathbf{S},\mathbf{T}}}$ in general depends on the choice of \mathbf{S} , and \mathbf{T} . However, the key point is that, if \mathbf{S}' and \mathbf{T}' is another pair of subsets satisfying similar conditions and $\#\mathbf{S}_\infty - 2\#\mathbf{T} = \#\mathbf{S}'_\infty - 2\#\mathbf{T}'$ (which will be the case we consider later in this paper), then the subgroup $\mathcal{E}_{G_{\mathbf{S},\mathbf{T}}}$ is the same as $\mathcal{E}_{G_{\mathbf{S}',\mathbf{T}'}}$.

Remark 2.4. Using Proposition 2.10 and Construction 2.15 later, we can access to most of the statements in [TX13⁺a] which were initially proved for unitary groups and interpreted using connected Shimura varieties. The key point mentioned in Notation 2.3 has the additional benefit that the description of the Goren–Oort strata actually descend to quaternionic Shimura varieties because now the subgroups $\mathcal{E}_{G_{\mathbf{S},\mathbf{T}}}$ ’s are compatible for different \mathbf{S} and \mathbf{T} ’s.

2.5. Automorphic representations. Following [TX13⁺b, §5.10], we use $\mathcal{A}_{(k,w)}$ to denote the set of irreducible automorphic cuspidal representations π of $\mathrm{GL}_2(\mathbb{A}_F)$ whose archimedean component π_τ for each $\tau \in \Sigma_\infty$ is a discrete series of weight $k_\tau - 2$ with central character $x \mapsto x^{w-2}$, and whose \mathfrak{p} -component $\pi_{\mathfrak{p}}$ is *unramified*. We write $\pi^{\infty,\mathfrak{p}}$ to denote the prime-to- \mathfrak{p} finite part of π .

We denote by $\rho_\pi : \mathrm{Gal}_F \rightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ the Galois representation attached to π normalized so that if v is a finite place of F at which π is unramified and K is hyperspecial, the action of a *geometric* Frobenius at v has trace equal to the eigenvalue of usual Hecke operator T_v on $(\pi^\infty)^K$. Let $\rho_{\pi,\mathfrak{p}}$ be the restriction of ρ_π to $\mathrm{Gal}_{\mathbb{F}_{p^g}}$ (note that ρ_π is unramified at \mathfrak{p}).

If $K_p = \mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}})$, the Hecke operators $T_{\mathfrak{p}}$ and $S_{\mathfrak{p}}$ are given by the action on $\pi_{\mathfrak{p}}^{K_p}$ of the double cosets $K_p \begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix} K_p$ and $K_p p^{-1} K_p$, respectively. Then, for $\pi \in \mathcal{A}_{(k,w)}$, the characteristic polynomial of $\rho_{\pi,\mathfrak{p}}(\mathrm{Frob}_{p^g})$ is given by

$$(2.5.1) \quad X^2 - T_{\mathfrak{p}}(\pi)X + S_{\mathfrak{p}}(\pi)p^g,$$

where $T_{\mathfrak{p}}(\pi)$ and $S_{\mathfrak{p}}(\pi)$ denote respectively the eigenvalues of $T_{\mathfrak{p}}$ and $S_{\mathfrak{p}}$ on $\pi_{\mathfrak{p}}^{K_p}$.

We say an automorphic representation $\pi \in \mathcal{A}_{(k,w)}$ *appears* in the cohomology of the Shimura variety $\mathrm{Sh}_K(G_{\mathbf{S},\mathbf{T}})$ if for each finite place v of \mathbf{S} , the local component π_v of π is square integrable so that π is the image, under the Jacquet–Langlands correspondence, of a unique automorphic representation $\pi_{B_{\mathbf{S}}}$ of $G_{\mathbf{S},\mathbf{T}}(\mathbb{A}) = (B_{\mathbf{S}} \otimes_{\mathbb{Q}} \mathbb{A})^\times$, and $(\pi_{B_{\mathbf{S}}}^\infty)^K$ is nonzero.

Notation 2.6. For $\pi \in \mathcal{A}_{(k,w)}$ and a $\overline{\mathbb{Q}}_\ell[G(\mathbb{A}^{\infty,p})]$ -module M , we write

$$M[\pi] := \mathrm{Hom}_{\overline{\mathbb{Q}}_\ell[G(\mathbb{A}^{\infty,p})]}(\pi_{B_{\mathbf{S}}}^{\infty,\mathfrak{p}}, M)$$

for its π -isotypical component. By the strong multiplicity one Theorem for quaternion algebras, $\pi_{B_{\mathbf{S}}}$ is determined by $\pi_{B_{\mathbf{S}}}^{\infty,\mathfrak{p}}$; this justifies the notation for $M[\pi]$. There is also a finite version: let $K^p \subset G(\mathbb{A}^{\infty,p})$ be an open compact subgroup so that $(\pi_{B_{\mathbf{S}}}^{\infty,\mathfrak{p}})^{K^p}$ is an irreducible module over the prime-to- p Hecke algebra $\mathbb{T}(K^p) := \overline{\mathbb{Q}}_\ell[K^p \backslash G(\mathbb{A}^{\infty,p})/K^p]$, and $\pi_{B_{\mathbf{S}}}$ is determined by the $\mathbb{T}(K^p)$ -module $\pi_{B_{\mathbf{S}}}^{\infty,\mathfrak{p}}$. Then, for a $\mathbb{T}(K^p)$ -module M , we put

$$M[\pi] := \mathrm{Hom}_{\mathbb{T}(K^p)}((\pi_{B_{\mathbf{S}}}^{\infty,\mathfrak{p}})^{K^p}, M).$$

2.7. An auxiliary CM field. To use the results in [TX13⁺a], we fix a CM extension E/F such that

- every place in \mathbf{S} is inert in E/F , and
- the place \mathfrak{p} splits as $\mathfrak{q}\bar{\mathfrak{q}}$ in E/F if $\#\mathbf{S}_\infty^c$ is even, and it is inert in E/F if $\#\mathbf{S}_\infty^c$ is odd.

These conditions imply that $B_{\mathbf{S}}$ splits over E . In later applications, we will need to consider several \mathbf{S} at the same time. We remark that, for all subsets \mathbf{S} involved later, the finite places contained in

\mathbf{S} are the same, and $\#\mathbf{S}_\infty^c$ will have the same parity. In particular, this means that we can fix for the rest of this paper one CM field E that satisfies the above conditions (for the initial $B_{\mathbf{S}}$).

We shall frequently use the following two finite idele elements:

- (1) \underline{p}_F denotes the finite idele in \mathbb{A}_F^∞ which is p at \mathfrak{p} and is 1 elsewhere (which we have already introduced in 2.1);
- (2) when \mathfrak{p} splits into $\mathfrak{q}\bar{\mathfrak{q}}$ in E , \underline{q} denotes the finite idele in \mathbb{A}_E^∞ which is p at \mathfrak{q} , p^{-1} at $\bar{\mathfrak{q}}$, and 1 elsewhere.

Let $\Sigma_{E,\infty}$ denote the set of complex embeddings of E . We fix a choice of subset $\tilde{\mathbf{S}}_\infty \subseteq \Sigma_{E,\infty}$ such that the natural restriction map $\Sigma_{E,\infty} \rightarrow \Sigma_\infty$ induces an isomorphism $\tilde{\mathbf{S}}_\infty \xrightarrow{\sim} \mathbf{S}_\infty$. When \mathfrak{p} splits into $\mathfrak{q}\bar{\mathfrak{q}}$, we use $\tilde{\mathbf{S}}_{\infty/\mathfrak{q}}$ (resp. $\tilde{\mathbf{S}}_{\infty/\bar{\mathfrak{q}}}$) to denote the subset of places in $\tilde{\mathbf{S}}_\infty$ inducing \mathfrak{q} (resp. $\bar{\mathfrak{q}}$) through the isomorphism ι_p . We put

$$(2.7.1) \quad \Delta_{\tilde{\mathbf{S}}} = \#\tilde{\mathbf{S}}_{\infty/\bar{\mathfrak{q}}} - \#\tilde{\mathbf{S}}_{\infty/\mathfrak{q}}.$$

We remark that all the $\tilde{\mathbf{S}}$'s we encounter later in this paper will all have the same $\Delta_{\tilde{\mathbf{S}}}$.

We write $E_{\mathfrak{p}}$ for $F_{\mathfrak{p}} \otimes_F E$. It is the quadratic unramified extension of $F_{\mathfrak{p}}$ if \mathfrak{p} is inert and it is $E_{\mathfrak{q}} \times E_{\bar{\mathfrak{q}}}$ if \mathfrak{p} splits. We set $\mathcal{O}_{E_{\mathfrak{p}}} := \mathcal{O}_{\mathfrak{p}} \otimes_{\mathcal{O}_F} \mathcal{O}_E$.

We put $\tilde{\mathbf{S}} = (\mathbf{S}, \tilde{\mathbf{S}}_\infty)$. Put $T_{E,\tilde{\mathbf{S}},\mathbf{T}} = T_E = \text{Res}_{E/\mathbb{Q}}(\mathbb{G}_m)$, where the subscript $(\tilde{\mathbf{S}}, \mathbf{T})$ means that we take the following Deligne homomorphism

$$h_{E,\tilde{\mathbf{S}},\mathbf{T}}: \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times \longrightarrow T_{E,\tilde{\mathbf{S}},\mathbf{T}}(\mathbb{R}) = \bigoplus_{\tau \in \Sigma_\infty} (E \otimes_{F,\tau} \mathbb{R}) \cong (\mathbb{C}^\times)^{\mathbf{S}_\infty - \mathbf{T}} \times (\mathbb{C}^\times)^{\mathbf{T}} \times (\mathbb{C}^\times)^{\mathbf{S}_\infty^c}$$

$$z = x + y\mathbf{i} \longmapsto \left((\bar{z}, \dots, \bar{z}), (z^{-1}, \dots, z^{-1}), (1, \dots, 1) \right).$$

Here, the isomorphism $E \otimes_{F,\tau} \mathbb{R} \simeq \mathbb{C}$ for $\tau \in \mathbf{S}_\infty$ is given by the chosen embedding $\tilde{\tau} \in \tilde{\mathbf{S}}_\infty$ lifting τ . One has the system of zero-dimensional Shimura varieties $\mathcal{Sh}_{K_E}(T_{E,\tilde{\mathbf{S}},\mathbf{T}})$ with \mathbb{C} -points given by:

$$\mathcal{Sh}_{K_E}(T_{E,\tilde{\mathbf{S}},\mathbf{T}})(\mathbb{C}) = E^{\times, \text{cl}} \backslash T_{E,\tilde{\mathbf{S}},\mathbf{T}}(\mathbb{A}^\infty) / K_E,$$

for any open compact subgroup $K_E \subseteq T_{E,\tilde{\mathbf{S}},\mathbf{T}}(\mathbb{A}^\infty) \cong \mathbb{A}_E^{\infty, \times}$. We put $K_{E,p} = \mathcal{O}_{E,p}^\times \subseteq T_{E,\tilde{\mathbf{S}},\mathbf{T}}(\mathbb{Q}_p)$, and write $\mathcal{Sh}_{K_{E,p}}(T_{E,\tilde{\mathbf{S}},\mathbf{T}}) = \varprojlim_{K_E^p} \mathcal{Sh}_{K_E^p K_{E,p}}(T_{E,\tilde{\mathbf{S}},\mathbf{T}})$ as the inverse limit over all open compact subgroups $K_E^p \subseteq T_{E,\tilde{\mathbf{S}},\mathbf{T}}(\mathbb{A}^{\infty,p})$.

As in Notation 2.3, we identify $T_{E,\tilde{\mathbf{S}},\mathbf{T}}(\mathbb{A}^\infty)$ for all $\tilde{\mathbf{S}}$ and \mathbf{T} , and write $T_E(\mathbb{A}^\infty)$ for it; so K_E is naturally its subgroup.

Under the isomorphism $\iota_p: \mathbb{C} \cong \overline{\mathbb{Q}}_p$, the image of the reflex field of $\mathcal{Sh}_{K_E}(T_{E,\tilde{\mathbf{S}},\mathbf{T}})$ is contained in $\mathbb{Q}_{p^{2g}}$. It makes sense to talk about $\mathcal{Sh}_{K_E}(T_{E,\tilde{\mathbf{S}},\mathbf{T}})_{\mathbb{Q}_{p^{2g}}}$. As $K_{E,p}$ is hyperspecial, the action of $\text{Gal}_{\mathbb{Q}_{p^{2g}}}$ on $\mathcal{Sh}_{K_E}(T_{E,\tilde{\mathbf{S}},\mathbf{T}})(\overline{\mathbb{Q}}_p)$ is unramified. So $\mathcal{Sh}_{K_E}(T_{E,\tilde{\mathbf{S}},\mathbf{T}})_{\mathbb{Q}_{p^{2g}}}$ is the disjoint union of the spectra of some finite unramified extension of $\mathbb{Q}_{p^{2g}}$, and it has an integral canonical model over $\mathbb{Z}_{p^{2g}}$ by taking the spectra of the corresponding rings of integers. Denote by $\text{Sh}_{K_E}(T_{E,\tilde{\mathbf{S}},\mathbf{T}})$ its special fiber. By Shimura's reciprocity law, the action of the geometric Frobenius $\text{Frob}_{p^{2g}}$ of $\mathbb{F}_{p^{2g}}$ on $\text{Sh}_{K_E}(T_{E,\tilde{\mathbf{S}},\mathbf{T}})(\overline{\mathbb{F}}_p)$ is given by

- (i) when \mathfrak{p} is inert in E/F , multiplication by $(\underline{p}_F)^{\#(\mathbf{S}_\infty - \#\mathbf{T}) - \#\mathbf{T}} = (\underline{p}_F)^{\#\mathbf{S}_\infty - 2\#\mathbf{T}}$ and
- (ii) when \mathfrak{p} splits into $\mathfrak{q}\bar{\mathfrak{q}}$, multiplication by

$$\frac{\varpi_{\mathfrak{q}}^{2(\#\tilde{\mathbf{S}}_{\infty/\bar{\mathfrak{q}}} - \#\mathbf{T})}}{\varpi_{\bar{\mathfrak{q}}}^{2(\#\tilde{\mathbf{S}}_{\infty/\mathfrak{q}} - \#\mathbf{T})}} = (\underline{p}_F)^{\#\mathbf{S}_\infty - 2\#\mathbf{T}}(\underline{q})^{\Delta_{\tilde{\mathbf{S}}}},$$

where $\varpi_{\mathfrak{q}}$ (resp. $\varpi_{\bar{\mathfrak{q}}}$) is the finite idele in \mathbb{A}_E^∞ which is p at the place \mathfrak{q} (resp. $\bar{\mathfrak{q}}$) and is 1 elsewhere, \underline{q} is the idele defined in Subsection 2.7(2) above, and $\Delta_{\tilde{\mathbf{S}}}$ is defined in (2.7.1).

In particular, if $(\tilde{S}', \mathbf{T}')$ is another pair above such that $\#\mathbf{S}'_\infty - 2\#\mathbf{T}' = \#\mathbf{S}_\infty - 2\#\mathbf{T}$ and $\Delta_{\tilde{S}} = \Delta_{\tilde{S}'}$ if \mathfrak{p} splits, then there exists an isomorphism of Shimura varieties over $\mathbb{F}_{p^{2g}}$:

$$(2.7.2) \quad \mathrm{Sh}_{K_E}(T_{E,\tilde{S},\mathbf{T}}) \xrightarrow{\cong} \mathrm{Sh}_{K_E}(T_{E,\tilde{S}',\mathbf{T}'})$$

compatible with the Hecke action of $T_E(\mathbb{A}^{\infty,p})$ on both sides as K_E^p varies.

2.8. A unitary Shimura variety. Let $Z = \mathrm{Res}_{F/\mathbb{Q}}(\mathbb{G}_m)$ be the center of $G_{\mathbf{S},\mathbf{T}}$. Put $G_{\tilde{S}}'' = G_{\mathbf{S},\mathbf{T}} \times_Z T_{E,\tilde{S},\mathbf{T}}$, which is the quotient of $G_{\mathbf{S},\mathbf{T}} \times T_{E,\tilde{S},\mathbf{T}}$ by Z embedded anti-diagonally as $z \mapsto (z, z^{-1})$. Consider the product Deligne homomorphism

$$h_{\mathbf{S},\mathbf{T}} \times h_{E,\tilde{S},\mathbf{T}}: \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times \rightarrow (G_{\mathbf{S},\mathbf{T}} \times T_{E,\tilde{S},\mathbf{T}})(\mathbb{R}),$$

which can be further composed with the quotient map to $G_{\tilde{S}}''$ to get

$$h_{\tilde{S}}'': \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times \rightarrow (G_{\mathbf{S},\mathbf{T}} \times_Z T_{E,\tilde{S},\mathbf{T}})(\mathbb{R}) \cong G_{\tilde{S}}''(\mathbb{R}).$$

Note that $h_{\tilde{S}}''$ does not depend on the choice of $\mathbf{T} \subseteq \mathbf{S}_\infty$ (hence the notation), and its conjugacy class is identified with $\mathfrak{H}_{\tilde{S}} = (\mathfrak{h}^\pm)^{\mathbf{S}_\infty}$. Let K_p'' denote the (maximal) open compact subgroup $\mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}}) \times_{\mathcal{O}_{\mathfrak{p}}^\times} \mathcal{O}_{E,\mathfrak{p}}^\times$ of $G_{\tilde{S}}''(\mathbb{Q}_p)$. We will consider open compact subgroups of the form $K'' = K_p'' K''^p \subset G_{\tilde{S}}''(\mathbb{A}^\infty)$ with $K''^p \subset G_{\tilde{S}}''(\mathbb{A}^{\infty,p})$. These data give rise to a Shimura variety $\mathrm{Sh}_{K''}(G_{\tilde{S}}'')$ (defined over $\mathbb{Q}_{p^{2g}}$), whose \mathbb{C} -points (via ι_p) are given by

$$\mathrm{Sh}_{K''}(G_{\tilde{S}}'')(\mathbb{C}) = G_{\tilde{S}}''(\mathbb{Q}) \backslash (\mathfrak{H}_{\tilde{S}} \times G_{\tilde{S}}''(\mathbb{A}^\infty)) / K''.$$

We put $\mathrm{Sh}_{K_p''}(G_{\tilde{S}}'') := \varprojlim_{K''^p} \mathrm{Sh}_{K''}(G_{\tilde{S}}'')$. Its geometric connected components admit a natural map

$$(2.8.1) \quad \pi_0(\mathrm{Sh}_{K_p''}(G_{\tilde{S}}'')_{\overline{\mathbb{Q}_p}}) \longrightarrow (F_+^{\times,\mathrm{cl}} \backslash \mathbb{A}_F^{\infty,\times} / \mathcal{O}_{\mathfrak{p}}^\times) \times (E^{\times,N_{E/F}=1,\mathrm{cl}} \backslash \mathbb{A}_E^{\infty,N_{E/F}=1} / \mathcal{O}_{E_{\mathfrak{p}}}^{N_{E/F}=1}).$$

As in the quaternionic case, this is an isomorphism if $\mathbf{S}_\infty^c \neq \emptyset$.

We write $\mathrm{Sh}_{K_p''}(G_{\tilde{S}}'')_{\overline{\mathbb{Q}_p}}^\circ$ for the preimage of $\mathbf{1} \times \mathbf{1}$, and call it *the neutral geometric connected component* of the unitary Shimura variety.

We can define the group $\mathcal{E}_{G_{\tilde{S}}''}$ and $\mathcal{G}_{\tilde{S},p}''$ for the Shimura data $(G_{\tilde{S}}'', h_{\tilde{S}}'')$ as in Subsection 2.2 (see e.g. [TX13⁺a, §2.11] for the recipe). First, we spell out the Shimura reciprocity map

$$(2.8.2) \quad \mathrm{Rec}_p'': \mathrm{Gal}_{\mathbb{F}_{p^{2g}}} \longrightarrow (F_+^{\times,\mathrm{cl}} \backslash \mathbb{A}_F^{\infty,\times} / \mathcal{O}_{\mathfrak{p}}^\times) \times (E^{\times,N_{E/F}=1,\mathrm{cl}} \backslash \mathbb{A}_E^{\infty,N_{E/F}=1} / \mathcal{O}_{E_{\mathfrak{p}}}^{N_{E/F}=1}),$$

where $N_{E/F}$ is the norm from E to F . The first argument of $\mathrm{Rec}_p''(\mathrm{Frob}_{p^{2g}})$ is given by the square of (2.2.2) times the norm of the element in Subsection 2.7(i) or (ii), and the second argument of $\mathrm{Rec}_p''(\mathrm{Frob}_{p^{2g}})$ is given by the element in Subsection 2.7(i) or (ii) divided by its complex conjugate. Explicitly,

- when \mathfrak{p} is inert in E/F , $\mathrm{Rec}_p''(\mathrm{Frob}_{p^{2g}}) = (\underline{p}_F)^{2g} \times 1$, and
- when \mathfrak{p} splits in E/F , $\mathrm{Rec}_p''(\mathrm{Frob}_{p^{2g}}) = (\underline{p}_F)^{2g} \times (\underline{q})^{2\Delta_{\tilde{S}}}$.

We put $\mathcal{G}_{\tilde{S},p}'' = (G_{\tilde{S}}''(\mathbb{A}^{\infty,p}) / \mathcal{O}_{E,(p)}^{\times,\mathrm{cl}}) \times \mathrm{Gal}_{\mathbb{F}_{p^{2g}}}$ ¹⁴ and define $\mathcal{E}_{G_{\tilde{S}}''}$ to be its subgroup of pairs (x, σ) such that $\nu''(x)$ is equal to $\mathrm{Rec}_p''(\sigma)^{-1}$, where

$$\begin{aligned} \nu'' : G_{\tilde{S}}'' = G_{\mathbf{S},\mathbf{T}} \times_Z T_{E,\tilde{S},\mathbf{T}} &\longrightarrow \mathrm{Res}_{F/\mathbb{Q}}(\mathbb{G}_m) \times \mathrm{Res}_{E/\mathbb{Q}}(\mathbb{G}_m)^{N_{E/F}=1} \\ (g, t) &\longmapsto (\nu(g)N_{E/F}(t), t/\bar{t}) \end{aligned}$$

is the natural morphism from $G_{\tilde{S}}''$ to its maximal abelian quotient.

¹⁴As in the footnote to (2.2.3), we omitted the star product in the definition of this group comparing to [TX13⁺a, (2.11.3)] because the center $\mathrm{Res}_{E/\mathbb{Q}}(\mathbb{G}_m)$ of $G_{\tilde{S},\mathbf{T}}''$ has trivial first cohomology.

Remark 2.9. Similar to Notation 2.3, if S' is another subset of places of F containing the same finite places as S (together with a choice of \tilde{S}'_∞), then $G''_{\tilde{S}'}(\mathbb{A}^\infty)$ is isomorphic to $G''_{\tilde{S}}(\mathbb{A}^\infty)$. We fix such an isomorphism, and denote them uniformly as $G''(\mathbb{A}^\infty)$. Hence we naturally identify groups $G''_{\tilde{S},p}$ for different \tilde{S} 's.

When $\#S_\infty = \#S'_\infty$ and $\Delta_{\tilde{S}} = \Delta_{\tilde{S}'}$ if \mathfrak{p} splits in E/F , the subgroup $\mathcal{E}_{G''_{\tilde{S}'}} \subset \mathcal{G}''_{\tilde{S}',p}$ can be also identified with $\mathcal{E}_{G''_{\tilde{S}}} \subset \mathcal{G}''_{\tilde{S},p}$. Indeed, in this case the reciprocity map Rec_p'' for \tilde{S} and \tilde{S}' are the same.

Proposition 2.10. (1) *We have a canonical isomorphism $\mathcal{E}_{G_{S,T}} \cong \mathcal{E}_{G''_{\tilde{S}}}$, and $\text{Sh}_{K_p}(G_{S,T})^\circ_{\mathbb{Q}_p^{\text{ur}}}$ together with the action of $\mathcal{E}_{G_{S,T}}$ is isomorphic to $\text{Sh}_{K_p''}(G''_{\tilde{S}})^\circ_{\mathbb{Q}_p^{\text{ur}}}$ together with the action of $\mathcal{E}_{G''_{\tilde{S}}}$.*

(2) *The Shimura varieties $\text{Sh}_K(G_{S,T})$ (resp. $\text{Sh}_{K_p''}(G''_{\tilde{S}})$) admit integral canonical models over \mathbb{Z}_{p^g} (resp. over $\mathbb{Z}_{p^{2g}}$), and the connected Shimura variety $\text{Sh}_{K_p}(G_{S,T})^\circ_{\mathbb{Q}_p^{\text{ur}}} \cong \text{Sh}_{K_p''}(G''_{\tilde{S}})^\circ_{\mathbb{Q}_p^{\text{ur}}}$ admits a canonical integral model over \mathbb{Z}_p^{ur} .*

Proof. For (1), the case when $T = \emptyset$ is treated in [TX13⁺a]. In general, note that the sequence of morphisms

$$G''_{\tilde{S}} \leftarrow G_{S,T} \times T_{E,\tilde{S},T} \rightarrow G_{S,T}$$

is compatible with the associated Deligne homomorphism, and the conjugacy classes of Deligne homomorphisms into various algebraic groups defined above are canonically identified. Standard facts (e.g. [TX13⁺a, Corollary 2.17]) about Shimura varieties implies that the series of morphisms of Shimura varieties

$$\text{Sh}_{K_p''}(G''_{\tilde{S}}) \leftarrow \text{Sh}_{K_p}(G_{S,T}) \times_{\mathbb{Z}_{p^g}} \text{Sh}_{K_{E,p}}(T_{E,\tilde{S},T}) \rightarrow \text{Sh}_{K_p}(G_{S,T})$$

induce isomorphisms on the neutral connected components. Hence, by [TX13⁺a, Theorem 3.14], there exists an integral canonical model for $\text{Sh}_{K_p''}(G''_{\tilde{S}})$ over $\mathbb{Z}_{p^{2g}}$, and thus the neutral connected component $\text{Sh}_{K_p''}(G''_{\tilde{S}})^\circ \cong \text{Sh}_{K_p}(G_{S,T})^\circ$ admits an integral canonical model over \mathbb{Z}_p^{ur} . Applying $\times_{\mathcal{E}_{G_{S,T}}} \mathcal{G}_{S,T,p}$, the latter induces an integral canonical model of $\text{Sh}_{K_p}(G_{S,T})$ over \mathbb{Z}_p^{ur} , which descends to \mathbb{Z}_{p^g} (cf. [TX13⁺a, Corollary 2.17]). \square

Remark 2.11. Statement (2) of Proposition 2.10 is a consequence of much more general theory of Kisin [Ki10]. However, in this paper, we will need essentially this explicit relationship between the integral models of $\text{Sh}_K(G_{S,T})$ and those of $\text{Sh}_{K_p''}(G''_{\tilde{S}})$.

Notation 2.12. We use $\text{Sh}_{K_p}(G_{S,T})$, $\text{Sh}_{K_{E,p}}(T_{E,\tilde{S},T})$, $\text{Sh}_{K_p''}(G''_{\tilde{S}})$, ... to denote the integral model over \mathbb{Z}_{p^g} or $\mathbb{Z}_{p^{2g}}$ of the corresponding Shimura variety, and use systematically Roman letters to denote the special fibers of Shimura varieties:

$$\text{Sh}_{K_p}(G_{S,T})^\star_{\mathbb{F}_p} := \text{Sh}_{K_p}(G_{S,T})^\star_{\mathbb{Z}_p^{\text{ur}}} \times_{\mathbb{Z}_p^{\text{ur}}} \overline{\mathbb{F}_p}, \quad \text{and} \quad \text{Sh}_?(G_{S,T}) := \text{Sh}_?(G_{S,T}) \times_{\mathbb{Z}_{p^g}} \mathbb{F}_{p^g}.$$

for $? = K$ or K_p , and $\star = \circ$ or \emptyset , and

$$\text{Sh}_{K_{E,p}}(T_{E,\tilde{S},T})_{\mathbb{F}_{p^{2g}}} = \text{Sh}_{K_{E,p}}(T_{E,\tilde{S},T}) \otimes_{\mathbb{Z}_{p^{2g}}} \mathbb{F}_{p^{2g}}, \quad \text{Sh}_{K_p''}(G''_{\tilde{S}})_{\mathbb{F}_{p^{2g}}} := \text{Sh}_{K_p''}(G''_{\tilde{S}}) \otimes_{\mathbb{Z}_{p^{2g}}} \mathbb{F}_{p^{2g}}$$

and similarly with open compact subgroups $K_E = K_{E,p} K_E^p \subset T_E(\mathbb{A}^\infty)$ and $K'' = K_p'' K''^p \subset G''(\mathbb{A}^\infty)$. We put $\text{Sh}_{K_p}(G''_{\tilde{S}})^\circ_{\mathbb{F}_p} = \text{Sh}_{K_p}(G''_{\tilde{S}})^\circ_{\mathbb{Z}_p^{\text{ur}}} \times_{\mathbb{Z}_p^{\text{ur}}} \overline{\mathbb{F}_p}$.

2.13. Automorphic sheaves. We now study the automorphic sheaves on these Shimura varieties. Fix a prime $\ell \neq p$ and an isomorphism $\iota_\ell : \mathbb{C} \simeq \overline{\mathbb{Q}_\ell}$. By a *regular multiweight*, we mean a tuple $(k, w) \in \mathbb{Z}^{\Sigma_\infty} \times \mathbb{Z}$ such that $k_\tau \equiv w \pmod{2}$ and $k_\tau \geq 2$ for all $\tau \in \Sigma_\infty$. Consider the algebraic representation

$$\rho_{S,T}^{(k,w)} = \boxtimes_{\tau \in \Sigma_\infty} (\text{Sym}^{k_\tau-2}(\text{std}^\vee) \otimes \det^{\frac{k_\tau-w}{2}})$$

of $G_{\mathbf{S},\mathbf{T}} \times \mathbb{C} \cong \prod_{\tau \in \Sigma_\infty} \mathrm{GL}_2(\mathbb{C})$, where std is the standard representation of $\mathrm{GL}_2(\mathbb{C})$. As explained in [Mil90], we have an automorphic $\overline{\mathbb{Q}}_\ell$ -lisse sheaf $\mathcal{L}_{\mathbf{S},\mathbf{T}}^{(k,w)}$ on $\mathcal{S}h_{K_p}(G_{\mathbf{S},\mathbf{T}})$ associated to $\rho_{\mathbf{S},\mathbf{T}}^{(k,w)}$. Note that $\mathcal{L}_{\mathbf{S},\mathbf{T}}^{(k,w)}$ is pure of weight $(w-2)(\#\mathbf{S}_\infty^c + 2\#\mathbf{T})$.

We fix a section $\tilde{\Sigma} \subset \Sigma_{E,\infty}$ of the natural restriction map $\Sigma_{E,\infty} \rightarrow \Sigma_\infty$ (which is independent of the choices $\tilde{\mathbf{S}}_\infty$). Let (\underline{k}, w) be a regular multiweight. Consider the following one-dimensional representation of $T_{E,\tilde{\mathbf{S}},\mathbf{T}} \times_{\mathbb{Q}} \mathbb{C} \cong \prod_{\tilde{\tau} \in \tilde{\Sigma}} \mathbb{G}_{m,\tilde{\tau}} \times \mathbb{G}_{m,\bar{\tilde{\tau}}}$:

$$\rho_{E,\tilde{\Sigma}}^w = \otimes_{\tilde{\tau} \in \tilde{\Sigma}} x^{2-w} \circ \mathrm{pr}_{E,\tilde{\tau}},$$

where $\bar{\tilde{\tau}}$ is the complex conjugate embedding of $\tilde{\tau}$, $\mathrm{pr}_{E,\tilde{\tau}}$ is the projection to the $\tilde{\tau}$ -component, and x^{2-w} is the character of \mathbb{C}^\times given by raising to the $(2-w)$ th power. This representation gives rise to a lisse $\overline{\mathbb{Q}}_\ell$ -étale sheaf $\mathcal{L}_{E,\tilde{\mathbf{S}},\mathbf{T},\tilde{\Sigma}}^w$ pure of weight $(w-2)(\#\mathbf{S}_\infty - 2\#\mathbf{T})$ on $\mathcal{S}h_{K_{E,p}}(T_{E,\tilde{\mathbf{S}},\mathbf{T}})$. If $\mathrm{Sh}_{K_E}(T_{E,\tilde{\mathbf{S}}',\mathbf{T}'})$ is another Shimura variety with $\#\mathbf{S}'_\infty - 2\#\mathbf{T}' = \#\mathbf{S}_\infty - 2\#\mathbf{T}$ and $\gamma : \mathrm{Sh}_{K_E}(T_{E,\tilde{\mathbf{S}},\mathbf{T}}) \cong \mathrm{Sh}_{K_E}(T_{E,\tilde{\mathbf{S}}',\mathbf{T}'})$ is the isomorphism (2.7.2), then we have a natural isomorphism

$$(2.13.1) \quad \gamma^*(\mathcal{L}_{E,\tilde{\mathbf{S}}',\mathbf{T}',\tilde{\Sigma}}^w) \xrightarrow{\sim} \mathcal{L}_{E,\tilde{\mathbf{S}},\mathbf{T},\tilde{\Sigma}}^w.$$

Let $\alpha_{\mathbf{T}} : G_{\mathbf{S},\mathbf{T}} \times T_{E,\tilde{\mathbf{S}},\mathbf{T}} \rightarrow G''_{\tilde{\mathbf{S}}}$ denote the natural quotient morphism. We have the following diagram.

$$(2.13.2) \quad \begin{array}{ccc} \mathcal{S}h_{K_p}(G_{\mathbf{S},\mathbf{T}}) & \xleftarrow{\mathrm{pr}_1} \mathcal{S}h_{K_p}(G_{\mathbf{S},\mathbf{T}}) \times_{\mathbb{Z}_{p^g}} \mathcal{S}h_{K_{E,p}}(T_{E,\tilde{\mathbf{S}},\mathbf{T}}) & \xrightarrow{\alpha_{\mathbf{T}}} \mathcal{S}h_{K_p''}(G''_{\tilde{\mathbf{S}}}) \\ & \downarrow \mathrm{pr}_2 & \\ & \mathcal{S}h_{K_{E,p}}(T_{E,\tilde{\mathbf{S}},\mathbf{T}}) & \end{array}$$

By our definition, the tensor product representation $\rho_{\mathbf{S},\mathbf{T}}^{(k,w)} \otimes \rho_{E,\tilde{\Sigma}}^w$ of $G_{\mathbf{S},\mathbf{T}} \times T_{E,\tilde{\mathbf{S}},\mathbf{T}}$ factors through $G''_{\tilde{\mathbf{S}}}$. This defines a $\overline{\mathbb{Q}}_\ell$ -lisse sheaf $\mathcal{L}_{\tilde{\mathbf{S}},\tilde{\Sigma}}^{''(k,w)}$ on $\mathcal{S}h_{K_p''}(G''_{\tilde{\mathbf{S}}})$ such that we have a canonical isomorphism

$$(2.13.3) \quad \alpha_{\mathbf{T}}^*(\mathcal{L}_{\tilde{\mathbf{S}},\tilde{\Sigma}}^{''(k,w)}) \cong \mathrm{pr}_1^*(\mathcal{L}_{\mathbf{S},\mathbf{T}}^{(k,w)}) \otimes \mathrm{pr}_2^*(\mathcal{L}_{E,\tilde{\mathbf{S}},\mathbf{T},\tilde{\Sigma}}^w).$$

Put $D = B_{\tilde{\mathbf{S}}} \otimes_F E$. Then our choice of E/F in 2.7 implies that $D \cong \mathrm{M}_{2 \times 2}(E)$, which explains the omission of \mathbf{S} in our notation. We fix such an isomorphism, and take a maximal order $\mathcal{O}_D \cong \mathrm{M}_{2 \times 2}(\mathcal{O}_E)$. Recall that there exists a versal family of abelian varieties $a : \mathbf{A}_{\tilde{\mathbf{S}},K_p''}'' \rightarrow \mathcal{S}h_{K_p''}(G''_{\tilde{\mathbf{S}}})$ [TX13⁺a, 3.20] equipped with a natural action by \mathcal{O}_D . Here, “versal” means that the Kodaira–Spencer map for the family $\mathbf{A}_{\tilde{\mathbf{S}}}''$ is an isomorphism. Using $\mathbf{A}_{\tilde{\mathbf{S}}}''$, $\mathcal{L}_{\tilde{\mathbf{S}},\tilde{\Sigma}}^{''(k,w)}$ can be reinterpreted as follows. Put $H_\ell(\mathbf{A}_{\tilde{\mathbf{S}}}'') = R^1 a_*(\overline{\mathbb{Q}}_\ell)$, which is an ℓ -adic local system on $\mathcal{S}h_{K_p''}(G''_{\tilde{\mathbf{S}}})$ equipped with an induced action by $\mathrm{M}_{2 \times 2}(E)$. For each $\tilde{\tau} \in \Sigma_{E,\infty}$, let $H_\ell(\mathbf{A}_{\tilde{\mathbf{S}}}'')_{\tilde{\tau}}$ denote the direct summand of $H_\ell(\mathbf{A}_{\tilde{\mathbf{S}}}'')$ on which E acts via $E \xrightarrow{\tilde{\tau}} \mathbb{C} \xrightarrow{\iota_\ell} \overline{\mathbb{Q}}_\ell$. Consider the idempotent $\mathfrak{e} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{M}_{2 \times 2}(E)$. We put $H_\ell(\mathbf{A}_{\tilde{\mathbf{S}}}'')_{\tilde{\tau}}^\circ = \mathfrak{e} \cdot H_\ell(\mathbf{A}_{\tilde{\mathbf{S}}}'')_{\tilde{\tau}}$, which is an ℓ -adic local system on $\mathcal{S}h_{K_p''}(G''_{\tilde{\mathbf{S}}})$ of rank 2. We have a canonical decomposition

$$H_\ell(\mathbf{A}_{\tilde{\mathbf{S}}}'') = \bigoplus_{\tilde{\tau} \in \tilde{\Sigma}} (H_\ell(\mathbf{A}_{\tilde{\mathbf{S}}}'')_{\tilde{\tau}} \oplus H_\ell(\mathbf{A}_{\tilde{\mathbf{S}}}'')_{\bar{\tilde{\tau}}}) = \bigoplus_{\tilde{\tau} \in \tilde{\Sigma}} (H_\ell(\mathbf{A}_{\tilde{\mathbf{S}}}'')_{\tilde{\tau}}^{\circ, \oplus 2} \oplus H_\ell(\mathbf{A}_{\tilde{\mathbf{S}}}'')_{\bar{\tilde{\tau}}}^{\circ, \oplus 2}).$$

Then one has

$$(2.13.4) \quad \mathcal{L}_{\tilde{\mathbf{S}},\tilde{\Sigma}}^{''(k,w)} = \bigotimes_{\tilde{\tau} \in \tilde{\Sigma}} \left(\mathrm{Sym}^{k_{\tilde{\tau}}-2} H_\ell(\mathbf{A}_{\tilde{\mathbf{S}}}'')_{\tilde{\tau}}^\circ \otimes (\wedge^2 H_\ell(\mathbf{A}_{\tilde{\mathbf{S}}}'')_{\tilde{\tau}}^\circ)^{\frac{w-k_{\tilde{\tau}}}{2}} \right).$$

Remark 2.14. We shall introduce a general construction below to relate the unitary Shimura varieties and the quaternionic Shimura varieties. We point out beforehand that the entire construction is modeled on the following question: By Hilbert 90 theorem, we have an exact sequence:

$$1 \rightarrow F^{\times, \text{cl}} \backslash \mathbb{A}_F^{\infty, \times} / \mathcal{O}_{\mathfrak{p}}^{\times} \rightarrow E^{\times, \text{cl}} \backslash \mathbb{A}_E^{\infty, \times} / \mathcal{O}_{E_{\mathfrak{p}}}^{\times} \xrightarrow{z \mapsto z/\bar{z}} E^{\times, N_{E/F}=1, \text{cl}} \backslash \mathbb{A}_E^{\infty, N_{E/F}=1} / \mathcal{O}_{E_{\mathfrak{p}}}^{N_{E/F}=1} \rightarrow 1.$$

The construction involves picking a preimage of some element in the target of the surjective map above. In general, there is no canonical choice of this preimage, and all choices form a torsor under the group $F^{\times, \text{cl}} \backslash \mathbb{A}_F^{\infty, \times} / \mathcal{O}_{\mathfrak{p}}^{\times}$. In very special case when the element in the target of the surjective map is trivial, one can have a canonical choice of its preimage, namely the identity element 1.

Construction 2.15. We now discuss a very important process that allows us to transfer certain correspondences on the unitary Shimura varieties $\text{Sh}_{K_p''}(G_{\tilde{\mathbf{S}}}'')$ to the quaternionic Shimura varieties $\text{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})$. Throughout this subsection, we assume that we are given two sets of data: $\tilde{\mathbf{S}}, \mathbf{T}$, and $\tilde{\mathbf{S}}', \mathbf{T}'$ as above, and that they satisfy the following conditions:

$$(2.15.1) \quad \#\mathbf{S}_{\infty} - 2\#\mathbf{T} = \#\mathbf{S}'_{\infty} - 2\#\mathbf{T}', \quad \Delta_{\tilde{\mathbf{S}}} = \Delta_{\tilde{\mathbf{S}}'} \text{ if } \mathfrak{p} \text{ splits in } E/F,$$

and the finite places contained in \mathbf{S} and those in \mathbf{S}' are the same. By (2.7.2), this implies that the Shimura varieties $\text{Sh}_{K_{E,p}}(T_{E, \tilde{\mathbf{S}}, \mathbf{T}})$ and $\text{Sh}_{K_{E,p}}(T_{E, \tilde{\mathbf{S}}', \mathbf{T}'})$ are isomorphic.

Suppose that we are given a correspondence between the two unitary Shimura varieties

$$(2.15.2) \quad \text{Sh}_{K_p''}(G_{\tilde{\mathbf{S}}}'') \xleftarrow{\pi''} X \xrightarrow{\eta''} \text{Sh}_{K_p''}(G_{\tilde{\mathbf{S}}'}''),$$

where the group $\mathcal{G}_{\tilde{\mathbf{S}}, p}'' \cong \mathcal{G}_{\tilde{\mathbf{S}}', p}''$ acts on all three spaces and the morphisms are equivariant for the actions. We further assume that the fibers of π'' are geometrically connected.

Step I: We will complete the correspondence above into the following commutative diagram

$$(2.15.3) \quad \begin{array}{ccccc} \text{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}}) \times_{\mathbb{F}_{p^d}} \text{Sh}_{K_{E,p}}(T_{E, \tilde{\mathbf{S}}, \mathbf{T}}) & \xleftarrow{\pi^{\times}} & Y & \xrightarrow{\eta^{\times}} & \text{Sh}_{K_p}(G_{\mathbf{S}', \mathbf{T}'}) \times_{\mathbb{F}_{p^d}} \text{Sh}_{K_{E,p}}(T_{E, \tilde{\mathbf{S}}', \mathbf{T}'}) \\ \downarrow \alpha_{\mathbf{T}} & & \downarrow \alpha_{\mathbf{T}}'' & & \downarrow \alpha_{\mathbf{T}'}' \\ \text{Sh}_{K_p''}(G_{\tilde{\mathbf{S}}}'') & \xleftarrow{\pi''} & X & \xrightarrow{\eta''} & \text{Sh}_{K_p''}(G_{\tilde{\mathbf{S}}'}''), \end{array}$$

so that Y is defined as the Cartesian product of the left square, and the top line is equivariant for the actions of $\mathcal{G}_{\mathbf{S}, \mathbf{T}, p} \times \mathbb{A}_E^{\infty, \times} \cong \mathcal{G}_{\mathbf{S}', \mathbf{T}', p} \times \mathbb{A}_E^{\infty, \times}$. For this, it suffices to lift the morphism η'' to η^{\times} . We point out that both $\alpha_{\mathbf{T}}$ and $\alpha_{\mathbf{T}'}'$ map every geometric connected component isomorphically to another geometric connected component.

We now separate the discussion (but not in an essential way) depending on whether \mathbf{S}_{∞}^c is empty.

- When $\mathbf{S}_{\infty}^c \neq \emptyset$, let Y° denote the preimage $(\pi^{\times})^{-1}(\text{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})_{\mathbb{F}_p}^{\circ} \times \{\mathbf{1}\})$, where $\mathbf{1}$ denotes the neutral point, namely the image of $1 \in \mathbb{A}_E^{\infty, \times}$ in $\text{Sh}_{K_{E,p}}(T_{E, \tilde{\mathbf{S}}, \mathbf{T}})_{\mathbb{F}_p}$. By our assumption on π'' , Y° is a geometric connected component of Y . Its image under $\eta'' \circ \alpha_{\mathbf{T}}''$ lies in a geometric connected component of $\text{Sh}_{K_p''}(G_{\tilde{\mathbf{S}}}'')$, say $\text{Sh}_{K_p''}(G_{\tilde{\mathbf{S}}'}'')_{\mathbb{F}_p}^{\bullet}$, corresponding to some $(\mathbf{x}, \mathbf{s}) \in (F_+^{\times, \text{cl}} \backslash \mathbb{A}_F^{\infty, \times} / \mathcal{O}_{\mathfrak{p}}^{\times}) \times (E^{\times, N_{E/F}=1, \text{cl}} \backslash \mathbb{A}_E^{\infty, N_{E/F}=1} / \mathcal{O}_{E_{\mathfrak{p}}}^{N_{E/F}=1})$ via the map (2.8.1). By Hilbert's 90th Theorem, there exists $\mathbf{t} \in E^{\times, \text{cl}} \backslash \mathbb{A}_E^{\infty, \times} / \mathcal{O}_{E_{\mathfrak{p}}}^{\times}$ with $\mathbf{t}/\bar{\mathbf{t}} = \mathbf{s}$, and the choice of \mathbf{t} is unique up to $F^{\times, \text{cl}} \backslash \mathbb{A}_F^{\infty, \times} / \mathcal{O}_{F_{\mathfrak{p}}}^{\times}$. We claim that giving a $(\mathcal{G}_{\mathbf{S}, \mathbf{T}, p} \times \mathbb{A}_E^{\infty, \times})$ -equivariant map η^{\times} as above is *equivalent* to choosing such a \mathbf{t} .

Indeed, let $\text{Sh}_{K_p}(G_{\mathbf{S}', \mathbf{T}'})_{\mathbb{F}_p}^{\bullet}$ be the connected component of $\text{Sh}_{K_p}(G_{\mathbf{S}', \mathbf{T}'})_{\mathbb{F}_p}$ corresponding to $\mathbf{y} = \mathbf{x} N_{E/F}(\mathbf{t})^{-1}$ via the map (2.2.1). Then $\alpha_{\mathbf{T}'}'$ sends $\text{Sh}_{K_p}(G_{\mathbf{S}', \mathbf{T}'})_{\mathbb{F}_p}^{\bullet} \times \{\mathbf{t}\}$ isomorphically to $\text{Sh}_{K_p''}(G_{\tilde{\mathbf{S}}'}'')_{\mathbb{F}_p}^{\bullet}$. Note that Y (resp. $\text{Sh}_{K_p}(G_{\mathbf{S}', \mathbf{T}'}) \times_{\mathbb{F}_{p^d}} \text{Sh}_{K_{E,p}}(T_{E, \tilde{\mathbf{S}}', \mathbf{T}'})$) can be recovered

from Y° (resp. $\mathrm{Sh}_{K_p}(G_{\mathbf{s}',\mathbf{T}'}^\bullet)_{\mathbb{F}_p} \times \{\mathbf{t}\}$ for any $\mathbf{t} \in E^{\times,\mathrm{cl}} \backslash \mathbb{A}_E^{\infty,\times} / \mathcal{O}_{E_p}^\times$ ¹⁵) by applying $- \times_{\mathcal{E}_{G_{\mathbf{s},\mathbf{T}}}} (\mathcal{G}_{\mathbf{s},\mathbf{T},p} \times E^{\times,\mathrm{cl}} \backslash \mathbb{A}_E^{\infty,\times} / \mathcal{O}_{E_p}^\times)$. Here, recall that $\mathcal{E}_{G_{\mathbf{s},\mathbf{T}}} \cong \mathcal{E}_{G_{\mathbf{s}}''}$ by Proposition 2.10(1), and it embeds into the product $\mathcal{G}_{\mathbf{s},\mathbf{T},p} \times E^{\times,\mathrm{cl}} \backslash \mathbb{A}_E^{\infty,\times} / \mathcal{O}_{E_p}^\times$ as follows: the morphism from $\mathcal{E}_{G_{\mathbf{s},\mathbf{T}}}$ to the first factor is the natural embedding and that to the second factor is given by first projecting to the Galois factor and then applying the Shimura reciprocity map as specified in Subsection 2.7(i) and (ii). Therefore, once such a \mathbf{t} is chosen, we can define η^\times as the morphism obtained by applying $- \times_{\mathcal{E}_{G_{\mathbf{s},\mathbf{T}}}} (\mathcal{G}_{\mathbf{s},\mathbf{T},p} \times E^{\times,\mathrm{cl}} \backslash \mathbb{A}_E^{\infty,\times} / \mathcal{O}_{E_p}^\times)$ to the map

$$Y^\circ \xrightarrow{\eta'' \circ \alpha_{\mathbf{T}'}''} \mathrm{Sh}_{K_p''}(G_{\mathbf{s}'}'')_{\mathbb{F}_p}^\bullet \xrightarrow{\sim} \mathrm{Sh}_{K_p}(G_{\mathbf{s}',\mathbf{T}'}^\bullet)_{\mathbb{F}_p} \times \{\mathbf{t}\},$$

where the last isomorphism is the inverse of the restriction of $\alpha_{\mathbf{T}'}'$ to $\mathrm{Sh}_{K_p}(G_{\mathbf{s}',\mathbf{T}'}^\bullet)_{\mathbb{F}_p} \times \{\mathbf{t}\}$.

Conversely, it is also clear that such a \mathbf{t} is determined by η^\times .

- When $\mathbf{S}_\infty^c = \emptyset$, a slight rewording is needed. Let X° denote the preimage under π'' of the \mathbb{F}_p -point $\mathbf{1} \in \mathrm{Sh}_{K_p''}(G_{\mathbf{s}}'')_{\mathbb{F}_p}$. So it is mapped under η'' to a point $\mathbf{g}'' \in \mathrm{Sh}_{K_p''}(G_{\mathbf{s}}'')_{\mathbb{F}_p}$. Let Y° denote the preimage under π^\times of the \mathbb{F}_p -point $(\mathbf{1}, \mathbf{1}) \in \mathrm{Sh}_{K_p}(G_{\mathbf{s},\mathbf{T}})_{\mathbb{F}_p} \times \mathrm{Sh}_{K_{E,p}}(T_{E,\tilde{\mathbf{s}},\mathbf{T}})_{\mathbb{F}_p}$. Then the map η^\times must take Y° to an \mathbb{F}_p -point (\mathbf{x}, \mathbf{t}) in $\alpha_{\mathbf{T}'}'^{-1}(\mathbf{g}'')$, and conversely, η^\times is determined by this choice of such a point by the same argument as above using the fact that η^\times is equivariant for the $(\mathcal{G}_{\mathbf{s},\mathbf{T},p} \times \mathbb{A}_E^{\infty,\times})$ -action.

In summary, one can always define such a lift η^\times , depending on a choice of a certain element $\mathbf{t} \in E^{\times,\mathrm{cl}} \backslash \mathbb{A}_E^{\infty,\times} / \mathcal{O}_{E_p}^\times$ which is unique up to multiplication by an element of $F^{\times,\mathrm{cl}} \backslash \mathbb{A}_F^{\infty,\times} / \mathcal{O}_p^\times$. In this case, we say that η^\times is constructed with *shift* \mathbf{t} . In general, we do not have a canonical choice for \mathbf{t} , and hence neither for η^\times . However, in the special case when $\mathrm{Sh}_{K_p''}(G_{\mathbf{s}'}'')_{\mathbb{F}_p}^\bullet$ is the neutral connected component $\mathrm{Sh}_{K_p''}(G_{\mathbf{s}'}'')_{\mathbb{F}_p}^\circ$ in the former case and $\mathbf{g}'' = \mathbf{1}$ in the later case, there is a canonical choice of such lift, namely, the neutral connected component $\mathrm{Sh}_{K_p}(G_{\mathbf{s}',\mathbf{T}'}^\bullet)_{\mathbb{F}_p}^\circ \times \{\mathbf{1}\}$ in the former case and $(\mathbf{1}, \mathbf{1})$ in the latter case. So under this additional hypothesis, we do have a canonical map η^\times .

Step II: Suppose that we have constructed the diagram (2.15.3) with shift \mathbf{t} (which is canonical up to an element of $F^{\times,\mathrm{cl}} \backslash \mathbb{A}_F^{\infty,\times} / \mathcal{O}_p^\times$), we want to obtain a correspondence

$$(2.15.4) \quad \mathrm{Sh}_{K_p}(G_{\mathbf{s},\mathbf{T}})_{\mathbb{F}_{p^{2g}}} \xleftarrow{\pi} Z \xrightarrow{\eta} \mathrm{Sh}_{K_p}(G_{\mathbf{s}',\mathbf{T}'})_{\mathbb{F}_{p^{2g}}}.$$

For this, it suffices to construct (2.15.4) over \mathbb{F}_p which carries equivariant action of $\mathrm{Gal}_{\mathbb{F}_{p^{2g}}}$. Starting with the top row of (2.15.3), composing η^\times with multiplication by \mathbf{t}^{-1} (note that $\mathrm{Sh}_{K_{E,p}}(T_{E,\tilde{\mathbf{s}},\mathbf{T}})_{\mathbb{F}_p}$ is in fact a group scheme), we get a correspondence¹⁶

$$(2.15.5) \quad \mathrm{Sh}_{K_p}(G_{\mathbf{s},\mathbf{T}})_{\mathbb{F}_p} \times \mathrm{Sh}_{K_{E,p}}(T_{E,\tilde{\mathbf{s}},\mathbf{T}})_{\mathbb{F}_p} \xleftarrow{\pi^\times} Y \xrightarrow{\mathbf{t}^{-1} \circ \eta^\times} \mathrm{Sh}_{K_p}(G_{\mathbf{s}',\mathbf{T}'})_{\mathbb{F}_p} \times \mathrm{Sh}_{K_{E,p}}(T_{E,\tilde{\mathbf{s}},\mathbf{T}'})_{\mathbb{F}_p},$$

which respects the projection to $\mathrm{Sh}_{K_{E,p}}(T_{E,\tilde{\mathbf{s}},\mathbf{T}})_{\mathbb{F}_p} \xrightarrow{\sim} \mathrm{Sh}_{K_{E,p}}(T_{E,\tilde{\mathbf{s}},\mathbf{T}'})_{\mathbb{F}_p}$. Taking the fiber of (2.15.5) over $\mathbf{1}$ of $\mathrm{Sh}_{K_{E,p}}(T_{E,\tilde{\mathbf{s}},\mathbf{T}})_{\mathbb{F}_p}$ gives (2.15.4), but to descend we need to modify the Galois action above (so that the Galois action preserves the fiber over $\mathbf{1}$) as follows: we change the action of $\mathrm{Frob}_{p^{2g}}$ on (2.15.5) by further composing with a Hecke action given by $1 \times (\underline{p}_F)^{2\#\mathbf{T} - \#\mathbf{S}_\infty} \in G(\mathbb{A}^\infty) \times \mathbb{A}_E^{\infty,\times}$ if \mathfrak{p} is inert in E/F , and $1 \times (\underline{p}_F)^{2\#\mathbf{T} - \#\mathbf{S}_\infty}(\mathfrak{q})^{-\Delta_{\tilde{\mathbf{s}}}}$ if \mathfrak{p} splits in E/F . This way, we have constructed a new Galois action on the factor $\mathrm{Sh}_{K_{E,p}}(T_{E,\tilde{\mathbf{s}},\mathbf{T}})_{\mathbb{F}_p}$. By usual Galois descent, we get (2.15.4).

¹⁵We point out that $E^{\times,\mathrm{cl}} \backslash \mathbb{A}_E^{\infty,\times} / \mathcal{O}_{E_p}^\times$ is canonically isomorphic to $\mathcal{O}_{E,(p)}^{\times,\mathrm{cl}} \backslash \mathbb{A}_E^{\infty,p,\times}$.

¹⁶Once again, both \mathbf{t} and this correspondence depend on the choice of the geometric connected component, and are uniquely defined up to multiplication by an element of $\mathcal{O}_{F,(p)}^{\times,\mathrm{cl}} \backslash \mathbb{A}_F^{\infty,\infty,p}$.

Step III: we will obtain a sheaf version of the construction above, namely, if in addition, we are given an isomorphism of sheaves

$$(2.15.6) \quad \eta''^\# : \pi''^*(\mathcal{L}_{\tilde{S}, \tilde{\Sigma}}''^{(k,w)}) \xrightarrow{\cong} \eta''^*(\mathcal{L}_{\tilde{S}', \tilde{\Sigma}}''^{(k,w)}),$$

then we will construct an isomorphism of sheaves

$$(2.15.7) \quad \eta^\# : \pi^*(\mathcal{L}_{\mathbf{S}, \mathbf{T}}^{(k,w)}) \xrightarrow{\cong} \eta^*(\mathcal{L}_{\mathbf{S}', \mathbf{T}'}^{(k,w)}),$$

which again depends on the choice of \mathbf{t} in Step I. First, pulling back (2.15.6) along $\alpha_{\mathbf{T}}''$ in the commutative diagram (2.15.3), we get

$$\alpha_{\mathbf{T}}''^*(\eta''^\#) : (\pi^\times)^*(\alpha_{\mathbf{T}}''^*(\mathcal{L}_{\tilde{S}, \tilde{\Sigma}}''^{(k,w)})) \xrightarrow{\cong} (\eta^\times)^*(\alpha_{\mathbf{T}'}''^*(\mathcal{L}_{\tilde{S}', \tilde{\Sigma}}''^{(k,w)}))$$

Taking into account the isomorphism (2.13.3), we have

$$\alpha_{\mathbf{T}}''^*(\eta''^\#) : (\pi^\times)^*(\mathrm{pr}_1^*(\mathcal{L}_{\mathbf{S}, \mathbf{T}}^{(k,w)}) \otimes \mathrm{pr}_2^*(\mathcal{L}_{E, \tilde{S}, \mathbf{T}, \tilde{\Sigma}}^w)) \xrightarrow{\cong} (\eta^\times)^*(\mathrm{pr}_1^*(\mathcal{L}_{\mathbf{S}', \mathbf{T}'}^{(k,w)}) \otimes \mathrm{pr}_2^*(\mathcal{L}_{E, \tilde{S}', \mathbf{T}', \tilde{\Sigma}}^w)).$$

Composing this with the action of \mathbf{t}^{-1} , we get a morphism

$$(\pi^\times)^*(\mathrm{pr}_1^*(\mathcal{L}_{\mathbf{S}, \mathbf{T}}^{(k,w)}) \otimes \mathrm{pr}_2^*(\mathcal{L}_{E, \tilde{S}, \mathbf{T}, \tilde{\Sigma}}^w)) \xrightarrow{\cong} (\mathbf{t}^{-1} \circ \eta^\times)^*(\mathrm{pr}_1^*(\mathcal{L}_{\mathbf{S}', \mathbf{T}'}^{(k,w)}) \otimes \mathrm{pr}_2^*(\mathcal{L}_{E, \tilde{S}', \mathbf{T}', \tilde{\Sigma}}^w)).$$

Since we may also identify the sheaves $\mathcal{L}_{E, \tilde{S}, \mathbf{T}, \tilde{\Sigma}}^w$ with $\mathcal{L}_{E, \tilde{S}', \mathbf{T}', \tilde{\Sigma}}^w$ using (2.13.1), we may restrict the morphism above to the fiber over the neutral point $\mathbf{1}$ and get a morphism of sheaves (2.15.7) we want over $\overline{\mathbb{F}}_p$. (Once again, this morphism is unique up to multiplication with an element of $F^{\times, \mathrm{cl}} \backslash \mathbb{A}_F^{\infty, \times} / \mathcal{O}_{\mathbf{p}}^\times$.) To descend it back down to $\mathbb{F}_{p^{2g}}$, we modify the action of the Frobenius by composing it with a central Hecke action as in Step II above. This concludes the needed construction.

Step IV: Understand the ambiguity appeared in the construction. We call η the morphism associated to η'' with shift \mathbf{t} , where $\mathbf{t} \in E^{\times, \mathrm{cl}} \backslash \mathbb{A}_E^{\infty, \times} / \mathcal{O}_{E, \mathbf{p}}^\times$ is the element appeared in Step I, and is determined only up to multiplication by an element of $F^{\times, \mathrm{cl}} \backslash \mathbb{A}_F^{\infty, \times} / \mathcal{O}_{\mathbf{p}}^\times$. When $\mathrm{Sh}_{K_p''}(G_{\tilde{S}}'')^\bullet = \mathrm{Sh}_{K_p''}(G_{\tilde{S}}'')^\circ$ in Step I, we can take $\mathbf{t} = \mathbf{1}$ and we get a canonically determined η with shift $\mathbf{1}$.

Finally, let us mention where the choice made in Step I is specified later in this paper. In Subsection 3.7, we invoke this construction to define the Goren–Oort cycles; this is where the choice will be fixed. Moreover, this choice will retroactively determine the choice we make when applying this construction to define link morphisms in the earlier Subsection 2.21, whenever this subsection is quoted. The shift will allow us to keep track of the choices we made.

Remark 2.16. Suppose that we are given two correspondences as in Construction 2.15. Namely, we have

- subsets of $\tilde{S}_i, \mathbf{T}_i$ for $i = 1, 2, 3$ such that $\#\mathbf{S}_{i, \infty} - 2\#\mathbf{T}_i$, the subset of \mathbf{S}_i of finite places, and $\Delta_{\tilde{S}_i}$ are independent of i , and
- two $\mathcal{G}_{\tilde{S}_i, p}''$ -equivariant correspondences between Shimura varieties

$$\mathrm{Sh}_{K_p''}(G_{\tilde{S}_i}'') \xleftarrow{\pi_i''} X_i \xrightarrow{\eta_i''} \mathrm{Sh}_{K_p''}(G_{\tilde{S}_{i+1}}'')$$

with $i = 1, 2$ such that π_i'' is a fiber bundle with geometric connected fibers.

Then we can compose these two correspondences to get a correspondence

$$\mathrm{Sh}_{K_p''}(G_{\tilde{S}_1}'') \xleftarrow{\pi_3''} X_3 := X_1 \times_{\eta_1'', \mathrm{Sh}_{K_p''}(G_{\tilde{S}_2}''), \pi_2''} X_2 \xrightarrow{\eta_3''} \mathrm{Sh}_{K_p''}(G_{\tilde{S}_3}'').$$

Thus we may apply Construction 2.15 to get correspondences (π_1, η_1) and (π_2, η_2) on the quaternionic Shimura varieties:

$$(2.16.1) \quad \begin{array}{ccccc} & & \xrightarrow{\pi_3} & X_3 & \xrightarrow{\eta_3} \\ & \swarrow \pi_1 & & \nwarrow \eta_1 & \searrow \pi_2 \\ \mathrm{Sh}_{K_p}(G_{\mathbf{S}_1, \mathbf{T}_1}) & & X_1 & & X_2 \\ & \nwarrow \eta_1 & & \swarrow \pi_2 & \searrow \eta_2 \\ & & \mathrm{Sh}_{K_p}(G_{\mathbf{S}_2, \mathbf{T}_2}) & & \mathrm{Sh}_{K_p}(G_{\mathbf{S}_3, \mathbf{T}_3}), \end{array}$$

with shifts $\mathbf{t}_1, \mathbf{t}_2$. Then their composition $(\pi_3, \eta_3) = (\pi_2, \eta_2) \circ (\pi_1, \eta_1)$ is the correspondence of quaternionic Shimura varieties associated to (π_3'', η_3'') with shift $\mathbf{t}_1 \mathbf{t}_2$.

Conversely, if we apply Construction 2.15 to (π_i'', η_i'') to get three correspondences (π_i, η_i) for $i = 1, 2, 3$ such that $(\pi_3, \eta_3) = (\pi_2, \eta_2) \circ (\pi_1, \eta_1)$, then their shifts satisfy the equality $\mathbf{t}_3 = \mathbf{t}_1 \mathbf{t}_2$.

For the rest of this paper, we always assume $K_p = \mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}})$.

2.17. Hecke operators at p . In this subsection, we consider the case $\mathbf{S}_{\infty}^c = \emptyset$, namely when the Shimura varieties are discrete. We want to relate the Hecke operators at p for the unitary and quaternionic Shimura varieties in a manner similar as above. We assume that \mathfrak{p} splits in E/F which is the case we need encounter later.

Let $\mathrm{Iw}_p \subseteq \mathrm{GL}_2(\mathcal{O}_{\mathfrak{p}})$ denote the subgroup consisting of matrices which are upper triangular when modulo \mathfrak{p} . The discussion in this section is designed to cover this case and give an integral canonical model $\mathcal{S}h_{\mathrm{Iw}_p}(G_{\mathbf{S}, \mathbf{T}})$ of the Shimura variety with Iwahori level structure. We denote by $T_{\mathfrak{p}}$ the Hecke correspondence given by the following diagram:

$$(2.17.1) \quad \begin{array}{ccc} & \mathcal{S}h_{\mathrm{Iw}_p}(G_{\mathbf{S}, \mathbf{T}}) & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ \mathcal{S}h_{K_p}(G_{\mathbf{S}, \mathbf{T}}) & & \mathcal{S}h_{K_p}(G_{\mathbf{S}, \mathbf{T}}), \end{array}$$

where π_1 is the natural projection, and π_2 sends the double coset of $x \in G(\mathbb{A}^{\infty})$ to that of $x \begin{pmatrix} p^{-1} & 0 \\ 0 & 1 \end{pmatrix}$.

For the unitary Shimura variety, we have $G''(\mathbb{Q}_p) \cong \mathrm{GL}_2(F_{\mathfrak{p}}) \times_{F_{\mathfrak{p}}^{\times}} (E_{\mathfrak{q}}^{\times} \times E_{\bar{\mathfrak{q}}}^{\times})$ and we use Iw_p'' to denote the subgroup $\mathrm{Iw}_p \times_{\mathcal{O}_{\mathfrak{p}}^{\times}} (\mathcal{O}_{E_{\mathfrak{q}}}^{\times} \times \mathcal{O}_{E_{\bar{\mathfrak{q}}}}^{\times})$. Similarly, we have an integral model $\mathcal{S}h_{\mathrm{Iw}_p''}(G_{\mathbf{S}}'')$ of the unitary Shimura variety with this Iwahori level structure. The element $\gamma_{\mathfrak{q}}'' = ((p^{-1} \ 0; 0 \ 1), (1, p)) \in G''(\mathbb{Q}_p)$ gives rise to a Hecke operator $T_{\mathfrak{q}}$ corresponding to the double coset $K_p'' \gamma_{\mathfrak{q}}'' K_p''$. Geometrically, it is given by the the following diagram

$$(2.17.2) \quad \begin{array}{ccc} & \mathcal{S}h_{\mathrm{Iw}_p''}(G_{\mathbf{S}}'') & \\ \pi_1'' \swarrow & & \searrow \pi_2'' \\ \mathcal{S}h_{K_p''}(G_{\mathbf{S}}'') & & \mathcal{S}h_{K_p''}(G_{\mathbf{S}}''), \end{array}$$

where π_1'' is the natural projection, and π_2'' sends the double coset of $x \in G''(\mathbb{A}^{\infty})$ to that of $x \gamma_{\mathfrak{q}}''$.

In a language similar to the previous subsection (except that we cannot quote it directly because the morphism π'' therein would not have geometric connected fibers), we may phrase the relation between the Hecke correspondences $T_{\mathfrak{p}}$ and $T_{\mathfrak{q}}$ in terms of the following commutative diagram (with

T_q vertical on the left and T_p vertical on the right)

$$\begin{array}{ccccc}
\mathrm{Sh}_{K_p''}(G_{\mathbb{S}}'') & \xleftarrow{\alpha_T} & \mathrm{Sh}_{K_p}(G_{\mathbb{S},T}) \times \mathrm{Sh}_{K_{E,p}}(T_{E,\tilde{\mathbb{S}},T}) & \xleftarrow{\text{fiber over } \mathbf{1}} & \mathrm{Sh}_{K_p}(G_{\mathbb{S},T}) \\
\uparrow \pi_1'' & & \uparrow \text{natural} & & \uparrow \pi_1 \\
\mathrm{Sh}_{Iw_p''}(G_{\mathbb{S}}'') & \xleftarrow{\alpha_T} & \mathrm{Sh}_{Iw_p}(G_{\mathbb{S},T}) \times \mathrm{Sh}_{K_{E,p}}(T_{E,\tilde{\mathbb{S}},T}) & \xleftarrow{\text{fiber over } \mathbf{1}} & \mathrm{Sh}_{Iw_p}(G_{\mathbb{S},T}) \\
\downarrow \pi_2'' & & \downarrow x \mapsto x \left(\begin{pmatrix} p_F^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \varpi_{\bar{q}} \right) & & \downarrow \pi_2 \\
\mathrm{Sh}_{K_p''}(G_{\mathbb{S}}'') & \xleftarrow{\alpha_T} & \mathrm{Sh}_{K_p}(G_{\mathbb{S},T}) \times \mathrm{Sh}_{K_{E,p}}(T_{E,\tilde{\mathbb{S}},T}) & & \\
& & \downarrow x \mapsto x(1, \varpi_{\bar{q}}^{-1}) & & \\
& & \mathrm{Sh}_{K_p}(G_{\mathbb{S},T}) \times \mathrm{Sh}_{K_{E,p}}(T_{E,\tilde{\mathbb{S}},T}) & \xleftarrow{\text{fiber over } \mathbf{1}} & \mathrm{Sh}_{K_p}(G_{\mathbb{S},T}).
\end{array}$$

So we may view T_p as the correspondence associated to T_q in a similar fashion to the previous subsection, with shift $\varpi_{\bar{q}}^{-1} \in E^{\times, \mathrm{cl}} \backslash \mathbb{A}_E^{\infty, \times} / \mathcal{O}_{E_p}^{\times}$.

2.18. Links. We recall briefly the notion of links introduced in [TX13⁺a, §7]. We put $g = [F : \mathbb{Q}]$ points aligned equi-distantly on a section of a vertical cylinder, one point corresponding to an archimedean place in Σ_{∞} (also identified with a p -adic embedding of F via $\iota_p : \mathbb{C} \cong \overline{\mathbb{Q}_p}$) so that the Frobenius action is equivalent to shifting the points in one direction. For a subset \mathbb{S} of places of F as above, we label places in \mathbb{S}_{∞} by a *plus sign* and places in \mathbb{S}_{∞}^c by a *node*. We call the entire picture a *band* corresponding to \mathbb{S} . We often draw the picture in the 2-dimensional xy -plane by thinking of x -coordinate modulo g . We present the points $\tau_0, \dots, \tau_{g-1}$ on the x -axis with coordinates $x = 0, \dots, g-1$, such that the Frobenius shifts the points to the right by 1, and shifts τ_{g-1} back to τ_0 (by first shifting to $x = g$ and thinking of the x -coordinate modulo g). For example, if F has degree 6 over \mathbb{Q} and $\mathbb{S}_{\infty} = \{\tau_1, \tau_3, \tau_4\}$, then we draw the band as $\bullet + \bullet + + \bullet$.

Suppose that \mathbb{S}' is another set of places of F with even cardinality such that it contains exactly the same finite places of F as \mathbb{S} and satisfies $\#\mathbb{S}_{\infty} = \#\mathbb{S}'_{\infty}$. We put the band for \mathbb{S} above the band for \mathbb{S}' on the same cylinder. In the 2-dimensional picture, we draw the band for \mathbb{S} on the line $y = 1$ and the band for \mathbb{S}' on the line $y = -1$. For each of the nodes of \mathbb{S} , we draw a curve starting from it and go monotonically downwards linking to a node of \mathbb{S}' (and ignore the plus signs) such that all the curves do not intersect with each other. Such a graph is called a *link* $\eta : \mathbb{S} \rightarrow \mathbb{S}'$. Two links are considered the same if the curves can be continuously deformed to each other while keeping all curves from intersecting. We say a curve is *turning to the left* (resp. *right*) if it can be deformed into a smooth curve which has positive (resp. negative) tangent slopes in the 2-dimensional picture. The *displacement* of a curve in η is the number of points it travels to the right, that is the difference between the x -coordinates of the ending and starting points of the curve (adding an appropriate multiple of g according to the times the curve wraps around the cylinder). The displacement is negative if the curve turns to the left. The *total displacement* $v(\eta)$ is the sum of displacements of all curves. For example, if $g = 5$, $\mathbb{S}_{\infty} = \{\tau_1, \tau_3\}$ and $\mathbb{S}'_{\infty} = \{\tau_2, \tau_4\}$, the link given by

$$(2.18.1) \quad \eta = \begin{array}{c} \bullet + \bullet + \bullet \\ \text{---} \text{---} \text{---} \text{---} \text{---} \\ \bullet \bullet \bullet + \bullet \end{array}.$$

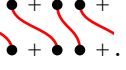

has total displacement $v(\eta) = 3 + 3 + 2 = 8$. For another example, the action of Frobenius σ twists the band and gives rise to a link $\sigma : \mathbb{S} \rightarrow \sigma(\mathbb{S})$ for which every curve is turning to the right with displacement 1, where $\sigma(\mathbb{S})$ is the set of places containing the same finite places as \mathbb{S} but $\sigma(\mathbb{S})_{\infty} = \sigma(\mathbb{S}_{\infty})$. Its total displacement is $v(\sigma) = d = \#\mathbb{S}_{\infty}^c$.

For a link $\eta : \mathbb{S} \rightarrow \mathbb{S}'$, we use $\eta^{-1} : \mathbb{S}' \rightarrow \mathbb{S}$ to denote the link obtained by flipping the picture about the equator of the cylinder. For two links $\eta : \mathbb{S} \rightarrow \mathbb{S}'$ and $\eta' : \mathbb{S}' \rightarrow \mathbb{S}''$, we have a natural

composition of links $\eta' \circ \eta : \mathbf{S} \rightarrow \mathbf{S}'$ given by putting the picture of η on top of the picture of η' and joint the nodes corresponding to \mathbf{S}' . It is obvious that $v(\eta^{-1}) = -v(\eta)$ and $v(\eta' \circ \eta) = v(\eta') + v(\eta)$.

When discussing the relative positions of two nodes of the band associated to \mathbf{S} , it is convenient to use the following

Notation 2.19. For $\tau \in \mathbf{S}_\infty^c$, let n_τ be the minimal positive integer such that $\sigma^{-n_\tau} \tau \in \mathbf{S}_\infty^c$. We put $\tau^- = \sigma^{-n_\tau} \tau$. We use τ^+ to denote the place in \mathbf{S}_∞^c such that $(\tau^+)^- = \tau$.

Example 2.20. A link from \mathbf{S} to itself can only be an integer power of the *fundamental link* $\eta_{\mathbf{S}}$,  that is to link each τ to τ^+ by shifting to the right with displacement n_{τ^+} . For example, . The total displacement of a fundamental link is exactly $v(\eta_{\mathbf{S}}) = g = [F : \mathbb{Q}]$.

2.21. Link morphisms I. Let \mathbf{S} and \mathbf{S}' be two even subsets of places of F consisting of the same finite places and $\#\mathbf{S}_\infty = \#\mathbf{S}'_\infty$. Let $\eta : \mathbf{S} \rightarrow \mathbf{S}'$ be a link such that all the curves (if there are any) are turning to the right. We allow the case $\mathbf{S}_\infty = \Sigma_\infty$ (so that there are no curves in the link η at all). In that case we say η is the *trivial link*. For each node $\tau \in \mathbf{S}_\infty^c$, we use $m(\tau)$ to denote the displacement of the curve connected to τ in η . Let $\tilde{\mathbf{S}}_\infty$ and $\tilde{\mathbf{S}}'_\infty$ be (any) lifts of \mathbf{S}_∞ and \mathbf{S}'_∞ as in Subsection 2.7. We have two unitary Shimura varieties $\mathrm{Sh}_{K_p''}(G_{\tilde{\mathbf{S}}}'')$ and $\mathrm{Sh}_{K_p''}(G_{\tilde{\mathbf{S}}'}'')$ as defined in 2.8.

We now recall the definition of the link morphism on $\mathrm{Sh}_{K_p''}(G_{\tilde{\mathbf{S}}}'')$ associated to η as introduced in [TX13⁺a, Definition 7.5]. Let n be an integer, which is always taken to be 0 if \mathfrak{p} is inert in E . A *link morphism* of *indentation degree* n associated to η on $\mathrm{Sh}_{K_p''}(G_{\tilde{\mathbf{S}}}'')$ is a pair $(\eta_{(n),\#}'', \eta_{(n),\#}^{\#\prime\prime})$, where

- (1) $\eta_{(n),\#}'' : \mathrm{Sh}_{K_p''}(G_{\tilde{\mathbf{S}}}'') \rightarrow \mathrm{Sh}_{K_p''}(G_{\tilde{\mathbf{S}}'}'')$ is a morphism of Shimura varieties that induces a bijection on geometric points;
- (2) $\eta_{(n),\#}^{\#\prime\prime} : \mathbf{A}_{\tilde{\mathbf{S}}}'' \rightarrow \eta_{(n),\#}^{\prime\prime*}(\mathbf{A}_{\tilde{\mathbf{S}}'}'')$ is a p -quasi-isogeny of abelian varieties compatible with the \mathcal{O}_D -actions, the polarizations, and the tame level structures;
- (3) for each geometric point x of $\mathrm{Sh}_{K_p''}(G_{\tilde{\mathbf{S}}}'')$ with image $x' = \eta_{(n),\#}''(x)$, if we write $\tilde{\mathcal{D}}(\mathbf{A}_{\tilde{\mathbf{S}},x}'')_{\tilde{\tau}}$ for the $\tilde{\tau}$ -component of the *covariant* Dieudonné module of $\mathbf{A}_{\tilde{\mathbf{S}},x}''$ for each $\tilde{\tau} \in \Sigma_{E,\infty}$, then there exists, for each $\tilde{\tau} \in \mathbf{S}_{E,\infty}^c$, some $t_{\tilde{\tau}} \in \mathbb{Z}$ *independent of the point* x such that

$$\eta_{(n),*}^{\#\prime\prime}(F_{\mathrm{es}, \mathbf{A}_{\tilde{\mathbf{S}},x}''}^{m(\tau)}(\tilde{\mathcal{D}}(\mathbf{A}_{\tilde{\mathbf{S}},x}'')_{\tilde{\tau}})) = p^{t_{\tilde{\tau}}} \tilde{\mathcal{D}}(\mathbf{A}_{\tilde{\mathbf{S}},x'}'')_{\sigma^{m(\tau)} \tilde{\tau}},$$

where $F_{\mathrm{es}, \mathbf{A}_{\tilde{\mathbf{S}},x}''}^{m(\tau)} : \tilde{\mathcal{D}}(\mathbf{A}_{\tilde{\mathbf{S}},x}'')_{\tilde{\tau}} \rightarrow \tilde{\mathcal{D}}(\mathbf{A}_{\tilde{\mathbf{S}},x}'')_{\sigma^{m(\tau)} \tilde{\tau}}$ is $m(\tau)$ -th iteration of the essential Frobenius for $\mathbf{A}_{\tilde{\mathbf{S}},x}''$ defined in [TX13⁺a, §4.2]; and

- (4) when \mathfrak{p} splits as $\mathfrak{q}\bar{\mathfrak{q}}$ in E , the degree of the quasi-isogeny

$$\eta_{(n),\mathfrak{q}}^{\#\prime\prime} : \mathbf{A}_{\tilde{\mathbf{S}}}''[\mathfrak{q}^\infty] \rightarrow \eta_{(n),\#}^{\prime\prime*}(\mathbf{A}_{\tilde{\mathbf{S}}'}''[\mathfrak{q}^\infty])$$

of the \mathfrak{q} -divisible groups is p^{2n} .

When the indentation degree n is clear by the context, we write simply $(\eta_{\#}'', \eta_{\#}^{\#\prime\prime})$ for $(\eta_{(n),\#}'', \eta_{(n),\#}^{\#\prime\prime})$.

For our purpose, the most important property we need is the following

Lemma 2.22 ([TX13⁺a], Proposition 7.8). *Let $\eta : \mathbf{S} \rightarrow \mathbf{S}'$ be a link as above. Then there exists at most one link morphism $(\eta_{(n),\#}'', \eta_{(n),\#}^{\#\prime\prime})$ with indentation degree n from $\mathrm{Sh}_{K_p''}(G_{\tilde{\mathbf{S}}}'')$ to $\mathrm{Sh}_{K_p''}(G_{\tilde{\mathbf{S}}'}'')$.*

Example 2.23. Let \mathbf{S} and $\tilde{\mathbf{S}}$ be as in the previous subsection. Let $\sigma^2 : \mathbf{S} \rightarrow \sigma^2(\mathbf{S})$ be the second iteration of the Frobenius link on \mathbf{S} . Put $\sigma^2 \tilde{\mathbf{S}} = (\sigma^2(\mathbf{S}), \sigma^2(\tilde{\mathbf{S}}_\infty))$. In [TX13⁺a, §3.22], we introduced natural morphisms called the *twisted (partial) Frobenius*

$$\mathfrak{F}_{\mathfrak{p}^2}'' : \mathrm{Sh}_{K_p''}(G_{\tilde{\mathbf{S}}}'') \rightarrow \mathrm{Sh}_{K_p''}(G_{\sigma^2 \tilde{\mathbf{S}}}'')$$

together with a quasi-isogeny of abelian varieties

$$\eta''_{\mathfrak{p}^2} : \mathbf{A}''_{\mathfrak{S}} \rightarrow (\mathfrak{F}''_{\mathfrak{p}^2})^* \mathbf{A}''_{\sigma^2 \mathfrak{S}}.$$

Such a morphism is characterized by the fact that the morphism $p\eta''_{\mathfrak{p}^2}$ is given by the p^2 -relative Frobenius. Then in the language of link morphism introduced above, $(\mathfrak{F}''_{\mathfrak{p}^2}, \eta''_{\mathfrak{p}^2})$ is the link morphism on $\mathrm{Sh}_{K_p''}(G''_{\mathfrak{S}})$ associated to the link $\eta = \sigma^2$ of indentation degree 0 if \mathfrak{p} is inert in E/F and of indentation degree $2\Delta_{\mathfrak{S}}$ if \mathfrak{p} splits in E/F ([TX13⁺a, Example 7.7(1)]).

Example 2.24. When \mathfrak{p} splits into $\mathfrak{q}\bar{\mathfrak{q}}$ in E/F , consider the Hecke operator $S_{\mathfrak{q}}$ given by multiplication by $1 \times \mathfrak{q}^{-1} \in G''(\mathbb{A}^{\infty}) = G(\mathbb{A}^{\infty}) \times_{\mathbb{A}_F^{\infty, \times}} \mathbb{A}_E^{\infty, \times}$ on the unitary Shimura variety. We start with the versal family of abelian varieties $\mathbf{A}''_{\mathfrak{S}}$ on $\mathrm{Sh}_{K_p''}(G''_{\mathfrak{S}})$, and put $\mathbf{B} := \mathbf{A}''_{\mathfrak{S}} \otimes_{\mathcal{O}_E} \bar{\mathfrak{q}} \cdot \mathfrak{q}^{-1}$ equipped with the induced action by \mathcal{O}_D . Let $\phi_{\mathfrak{q}} : \mathbf{A}''_{\mathfrak{S}} \rightarrow \mathbf{B}$ denote the natural p -quasi-isogeny induced by $\mathcal{O}_E \rightarrow \bar{\mathfrak{q}}\mathfrak{q}^{-1}$. We equip \mathbf{B} with the natural prime-to- p level structure compatible with $\phi_{\mathfrak{q}}$. The polarization $\lambda_{\mathbf{A}''_{\mathfrak{S}}}$ on $\mathbf{A}''_{\mathfrak{S}}$ naturally induces a polarization $\lambda_{\mathbf{B}}$ on \mathbf{B} such that $\lambda_{\mathbf{A}''_{\mathfrak{S}}} = \phi_{\mathfrak{q}}^{\vee} \circ \lambda_{\mathbf{B}} \circ \phi_{\mathfrak{q}}$. There is a unique morphism

$$S_{\mathfrak{q}} : \mathrm{Sh}_{K_p''}(G''_{\mathfrak{S}}) \rightarrow \mathrm{Sh}_{K_p''}(G''_{\mathfrak{S}}),$$

which, together with $\phi_{\mathfrak{q}}$, gives a link morphism for the trivial link $\mathrm{id} : \mathfrak{S} \rightarrow \mathfrak{S}$ of indentation degree $2g$.

If we apply Construction 2.15 to the morphism $S_{\mathfrak{q}}$ (with the morphism π there being trivial), we can lift it to a endomorphism of $\mathrm{Sh}_{K_p}(G_{\mathfrak{S}, \mathfrak{T}}) \times \mathrm{Sh}_{K_{E,p}}(T_{\mathfrak{S}, \mathfrak{T}})$ given by multiplication by $((p_F)^{-1}, \varpi_{\bar{\mathfrak{q}}}^{-2}) \in G(\mathbb{A}^{\infty}) \times_{\mathbb{A}_F^{\infty, \times}} \mathbb{A}_E^{\infty, \times}$. So the endomorphism $S_{\mathfrak{p}}$ given by multiplication by central element $(p_F)^{-1}$ may be viewed as the morphism on the quaternionic Shimura variety obtained by applying Construction 2.15 to the morphism $\eta'' = S_{\mathfrak{q}}$ with shift $\varpi_{\bar{\mathfrak{q}}}^2$.

2.25. Normalizations of link morphisms. Keep the notation of Subsection 2.21 and assume moreover that

- the link morphism $(\eta''_{(n), \#}, \eta''_{(n)}^{\#})$ on $\mathrm{Sh}_{K_p''}(G''_{\mathfrak{S}})$ exists,
- $\Delta_{\mathfrak{S}} = \Delta_{\mathfrak{S}'}$ if \mathfrak{p} splits in E/F , and
- we are given two subsets $\mathfrak{T} \subseteq \mathfrak{S}_{\infty}$ and $\mathfrak{T}' \subseteq \mathfrak{S}'_{\infty}$ such that $\#\mathfrak{T} = \#\mathfrak{T}'$.

Applying Construction 2.15 to the link morphism $(\eta''_{(n), \#}, \eta''_{(n)}^{\#})$ (with $X = \mathrm{Sh}_{K_p''}(G''_{\mathfrak{S}})$ and π'' in (2.15.2) equal to the identity), we get a pair of morphisms

$$\eta_{(n), \#} : \mathrm{Sh}_{K_p}(G_{\mathfrak{S}, \mathfrak{T}}) \rightarrow \mathrm{Sh}_{K_p}(G_{\mathfrak{S}', \mathfrak{T}'}), \quad \text{and} \quad \eta_{(n)}^{\#} : \mathcal{L}_{\mathfrak{S}, \mathfrak{T}}^{(k, w)} \xrightarrow{\cong} \eta_{(n), \#}^*(\mathcal{L}_{\mathfrak{S}', \mathfrak{T}'}^{(k, w)}),$$

with some shift $\mathbf{t} \in E^{\times, \mathrm{cl}} \backslash \mathbb{A}_E^{\infty, \times} / \mathcal{O}_{E, \mathfrak{p}}^{\times}$ (See Construction 2.15 Step IV). In the sequel, we call $(\eta_{(n), \#}, \eta_{(n)}^{\#})$ (or simply $\eta_{(n), \#}$ for short) *the link morphism with indentation degree n on the quaternionic Shimura variety $\mathrm{Sh}_{K_p}(G_{\mathfrak{S}, \mathfrak{T}})$ with shift \mathbf{t}* . Note that by Lemma 2.22 and Construction 2.15, for a fixed lifting $\tilde{\mathfrak{S}}$ of \mathfrak{S} , indentation degree n and shift \mathbf{t} , there exists at most one link morphism $(\eta_{(n), \#}, \eta_{(n)}^{\#})$ on $\mathrm{Sh}_K(G_{\mathfrak{S}, \mathfrak{T}})$.

The link morphism $(\eta_{(n), \#}, \eta_{(n)}^{\#})$ induces a homomorphism of the cohomology groups:

$$\begin{aligned} \tilde{\eta}_{(n)}^* : H_{\mathrm{et}}^*(\mathrm{Sh}_{K_p}(G_{\mathfrak{S}', \mathfrak{T}'}), \mathcal{L}_{\mathfrak{S}', \mathfrak{T}'}^{(k, w)}) &\longrightarrow H_{\mathrm{et}}^*(\mathrm{Sh}_{K_p}(G_{\mathfrak{S}, \mathfrak{T}}), \eta_{(n), \#}^*(\mathcal{L}_{\mathfrak{S}', \mathfrak{T}'}^{(k, w)})) \\ &\xrightarrow{(\eta_{(n), \#}^{\#})^{-1}} H_{\mathrm{et}}^*(\mathrm{Sh}_{K_p}(G_{\mathfrak{S}, \mathfrak{T}}), \mathcal{L}_{\mathfrak{S}, \mathfrak{T}}^{(k, w)}) \end{aligned}$$

which is equivariant under the Hecke action by $G(\mathbb{A}^\infty)$ ¹⁷ and the Galois action by $\text{Gal}_{\mathbb{F}_{p^{2g}}}$. We fix a square root $p^{1/2} \in \overline{\mathbb{Q}_\ell}$ of p . We put

$$(2.25.1) \quad \eta_{(n)}^* = \frac{1}{p^{v(\eta)/2}} \tilde{\eta}_{(n)}^*,$$

and call it the *normalized link morphism* on the cohomology groups of quaternionic Shimura varieties associated to η with indentation degree n and shift \mathbf{t} . This normalization will be justified in Lemma 2.29(2). We point out one more time that the (normalized) link morphism depends on the choice of the geometric connected component as in Construction 2.15 Step I, or equivalently the shift \mathbf{t} . When the link morphism $\eta_{(n),\sharp}'' : \text{Sh}_{K_p''}(G_{\mathbf{S}}'') \rightarrow \text{Sh}_{K_p''}(G_{\mathbf{S}'}'')$ preserves the neutral connected components, $\mathbf{t} = \mathbf{1}$ is a canonical choice, and in that case, $\eta_{(n)}^*$ is canonically defined.

Let $\eta_1 : \mathbf{S}_1 \rightarrow \mathbf{S}_2$ and $\eta_2 : \mathbf{S}_2 \rightarrow \mathbf{S}_3$ be two links with all curves turning to the right, satisfying the conditions above, i.e. all \mathbf{S}_i have the same set of finite places, $\#\mathbf{S}_{1,\infty} = \#\mathbf{S}_{2,\infty} = \#\mathbf{S}_{3,\infty}$, $\#\mathbf{T}_1 = \#\mathbf{T}_2 = \#\mathbf{T}_3$, and $\Delta_{\mathbf{S}_1} = \Delta_{\mathbf{S}_2} = \Delta_{\mathbf{S}_3}$ if \mathfrak{p} splits in E/F . Suppose that there are link morphisms $(\eta_{i,(n_i),\sharp}'', \eta_{i,(n_i)}'')$ for $i = 1, 2$ on unitary Shimura varieties with indentation degree n_i . Then the composed map

$$\eta_{12,(n_{12}),\sharp}'' : \text{Sh}_{K_p''}(G_{\mathbf{S}_1}'') \xrightarrow{\eta_{1,(n_1),\sharp}''} \text{Sh}_{K_p''}(G_{\mathbf{S}_2}'') \xrightarrow{\eta_{2,(n_2),\sharp}''} \text{Sh}_{K_p''}(G_{\mathbf{S}_3}'')$$

together with the composed quasi-isogeny

$$\eta_{12,(n_{12})}'' : \mathbf{A}_{\mathbf{S}_1}'' \xrightarrow{\eta_{1,(n_1)}''} \eta_{1,(n_1),\sharp}''(\mathbf{A}_{\mathbf{S}_2}'') \xrightarrow{\eta_{1,(n_1),\sharp}''(\eta_{2,(n_2)}'')} \eta_{1,(n_1),\sharp}'' \eta_{2,(n_2),\sharp}''(\mathbf{A}_{\mathbf{S}_3}'')$$

gives the (unique) link morphism on the unitary Shimura varieties with indentation degree $n_{12} = n_1 + n_2$ associated to the composed link $\eta_{12,(n_{12})}'' := \eta_{2,(n_2)}'' \circ \eta_{1,(n_1)}''$. From this, we get a link morphism of quaternionic Shimura varieties of indentation degree n_{12} :

$$\eta_{12,(n_{12}),\sharp} : \text{Sh}_{K_p}(G_{\mathbf{S}_1,\mathbf{T}_1}) \xrightarrow{\eta_{1,(n_1),\sharp}} \text{Sh}_{K_p}(G_{\mathbf{S}_2,\mathbf{T}_2}) \xrightarrow{\eta_{2,(n_2),\sharp}} \text{Sh}_{K_p}(G_{\mathbf{S}_3,\mathbf{T}_3}),$$

such that the shift of $\eta_{12,(n_{12}),\sharp}$ is the product of the shifts of $\eta_{1,(n_1),\sharp}$ and $\eta_{2,(n_2),\sharp}$. Moreover, we have $\eta_{12,(n_{12})}^* = \eta_{1,(n_1)}^* \circ \eta_{2,(n_2)}^*$ on the cohomology groups of quaternionic Shimura varieties.

Proposition 2.26. *Let $\pi \in \mathcal{A}_{(k,w)}$ be an automorphic representation appearing in the cohomology of the Shimura variety $\text{Sh}_K(G_{\mathbf{S},\mathbf{T}})$. Then we have*

$$(2.26.1) \quad H_{\text{et}}^i(\text{Sh}_K(G_{\mathbf{S},\mathbf{T}})_{\overline{\mathbb{F}_p}}, \mathcal{L}_{\mathbf{S},\mathbf{T}}^{(k,w)})[\pi] = \begin{cases} \rho_{\pi,\mathfrak{p}}^{\otimes d} \otimes [\det(\rho_{\pi,\mathfrak{p}})(1)]^{\otimes \#\mathbf{T}} & \text{if } i = d, \\ 0 & \text{if } i \neq d; \end{cases}$$

it is equivariant, up to semisimplification, for the action of the geometric Frobenius $\text{Frob}_{p^{2g}}$. Explicitly, if α_π, β_π are the two eigenvalues of $\rho_{\pi,\mathfrak{p}}(\text{Frob}_{p^g})$, then the (generalized) eigenvalues of the action of $\text{Frob}_{p^{2g}}$ on (2.26.1) are $p^{-2g\#\mathbf{T}} \alpha_\pi^{2(i+\#\mathbf{T})} \beta_\pi^{2(d-i+\#\mathbf{T})}$ with multiplicity $\binom{d}{i}$ for $0 \leq i \leq d$.

Proof. The first part of the proposition is well known to experts. We defer its proof to the Appendix (see Proposition A.3). The explicit description of the action of $\text{Frob}_{p^{2g}}$ is straightforward. \square

Proposition 2.27. *Assume that $d = \#\mathbf{S}_\infty^c \neq 0$.*

- (1) *The $2g$ -th iteration of the Frobenius link $\sigma^{2g} : \mathbf{S} \rightarrow \mathbf{S}$ coincides with the $2d$ -fold self-composition of the fundamental link $\eta_{\mathbf{S}}$ (2.20).*
- (2) *The link morphism on $\text{Sh}_{K_p''}(G_{\mathbf{S}}'')$ with indentation degree 0 associated to $\sigma^{2g} = \eta_{\mathbf{S}}^{2d}$ exists. It is given by*
 - (a) *g -fold self-composition $(\mathfrak{F}_{p^2}'')^g$ if \mathfrak{p} is inert in E/F ;*

¹⁷Here, $G(\mathbb{A}^\infty)$ denotes the common finite adelic points of $G_{\mathbf{S},\mathbf{T}}$ and $G_{\mathbf{S}',\mathbf{T}'}$ according to Notation 2.3.

(b) $(\mathfrak{F}_{\mathfrak{p}^2})^g \cdot S_q^{-\Delta_{\mathfrak{s}}}$ if \mathfrak{p} splits in E/F , where S_q is defined in Example 2.24.

Moreover, this link morphism preserves the neutral geometric connected component $\mathrm{Sh}_{K_p''}(G_{\mathfrak{s}}'')_{\mathbb{F}_p}^\circ$ and hence induces a canonical link morphism $(\eta_{\mathfrak{s},(0),\sharp}^{2d}, \eta_{\mathfrak{s},(0)}^{2d,\sharp})$ on the quaternionic Shimura variety $\mathrm{Sh}_{K_p}(G_{\mathfrak{s},\mathfrak{T}})$ with shift 1 for any fixed subset $\mathfrak{T} \subset S_\infty$.

(3) Let

$$(\eta_{\mathfrak{s}}^{2d})_{(0)}^\star : H_{\mathrm{et}}^d(\mathrm{Sh}_K(G_{\mathfrak{s},\mathfrak{T}})_{\mathbb{F}_p}, \mathcal{L}_{\mathfrak{s},\mathfrak{T}}^{(k,w)}) \rightarrow H_{\mathrm{et}}^d(\mathrm{Sh}_K(G_{\mathfrak{s},\mathfrak{T}})_{\mathbb{F}_p}, \mathcal{L}_{\mathfrak{s},\mathfrak{T}}^{(k,w)})$$

be the normalized link morphism (2.25.1) induced by $(\eta_{\mathfrak{s},(0),\sharp}^{2d}, \eta_{\mathfrak{s},(0)}^{2d,\sharp})$. Then we have an equality of operators on cohomology groups:

$$(2.27.1) \quad (\eta_{\mathfrak{s}}^{2d})_{(0)}^\star = p^{-dg} \cdot \mathrm{Frob}_{p^{2g}} \circ S_p^{-d-2\#\mathfrak{T}},$$

where S_p is the Hecke operator given by the central element $\underline{p}_F^{-1} \in G(\mathbb{A}^\infty)$. In particular, for each $\pi \in \mathcal{A}_{(k,w)}$ and each integer i with $0 \leq i \leq d$, the (generalized) eigenspace of $\mathrm{Frob}_{p^{2g}}$ on $H_{\mathrm{et}}^d(\mathrm{Sh}_K(G_{\mathfrak{s},\mathfrak{T}})_{\mathbb{F}_p}, \mathcal{L}_{\mathfrak{s},\mathfrak{T}}^{(k,w)})[\pi]$ with eigenvalue $p^{-2g\#\mathfrak{T}} \alpha_\pi^{2(i+\#\mathfrak{T})} \beta_\pi^{2(d-i+\#\mathfrak{T})}$ is the same as the (generalized) eigenspace of $(\eta_{\mathfrak{s}}^{2d})_{(0)}^\star$ with eigenvalue $(\alpha_\pi/\beta_\pi)^{2i-d}$.

Proof. Statement (1) is evident. For (2), we first check that the maps given by (a) and (b) are link morphisms with indentation degree 0 associated to the link $\eta_{\mathfrak{s}}^{2d}$. This follows easily from Examples 2.23 and 2.24. By the uniqueness of link morphisms (Lemma 2.22), they are the link morphisms we sought for.

We next show that the link morphism in the unitary case preserves the neutral geometric connected component $\mathrm{Sh}_{K_p''}(G_{\mathfrak{s}}'')_{\mathbb{F}_p}^\circ$. This is a direct computation using the Shimura reciprocity map (in Subsection 2.8), which we spell out now. Denote by Φ^{2g} the Frobenius endomorphism of $\mathrm{Sh}_{K_p''}(G_{\mathfrak{s}}'')$ relative to $\mathbb{F}_{p^{2g}}$. Then $(\mathfrak{F}_{\mathfrak{p}^2})^g$ is nothing but the composition of Φ^{2g} with the Hecke operator S_p^{-g} , where S_p is the Hecke correspondence given by the central element $(\underline{p}_F^{-1}, 1) \in G(\mathbb{A}^\infty) \times_{\mathbb{A}_F^\infty, \times} \mathbb{A}_E^{\infty, \times} \cong G'(\mathbb{A}^\infty)$. Recall that the set of geometric connected components of $\mathrm{Sh}_{K_p''}(G_{\mathfrak{s}}'')$ is given by

$$\pi_0(\mathrm{Sh}_{K_p''}(G_{\mathfrak{s}}'')_{\mathbb{F}_p}) \cong (F_+^{\times, \mathrm{cl}} \backslash \mathbb{A}_F^{\infty, \times} / \mathcal{O}_p^\times) \times (E^{\times, N_{E/F}=1, \mathrm{cl}} \backslash \mathbb{A}_E^{\infty, N_{E/F}=1} / \mathcal{O}_{E_p}^{N_{E/F}=1}).$$

The action of Φ^{2g} on $\pi_0(\mathrm{Sh}_{K_p''}(G_{\mathfrak{s}}'')_{\mathbb{F}_p})$ coincides with the arithmetic Frobenius $\mathrm{Frob}_{p^{2g}}^{-1} \in \mathrm{Gal}_{\mathbb{F}_{p^{2g}}}$, which is computed already by (2.8.2). We now list the actions of these operators on the geometric connected components.

Operator	When \mathfrak{p} splits	When \mathfrak{p} is inert
Φ^{2g}	$(\underline{p}_F)^{-2g} \times (\underline{q})^{-2\Delta_{\mathfrak{s}}}$	$(\underline{p}_F)^{-2g} \times 1$
S_p	$(\underline{p}_F)^{-2} \times 1$	$(\underline{p}_F)^{-2} \times 1$
S_q	$1 \times \underline{q}^{-2}$	N/A

It is now clear that the link morphisms given in (1) and (2) preserve the neutral geometric connected component. This verifies (2).

We now turn to the proof of (3). It suffices to verify (2.27.1) because S_p acts on the π -component by the scalar $\omega_\pi(\underline{p}^{-1}) = \alpha_\pi \beta_\pi / p^g$ according to (2.5.1), and then statement (3) would follow immediately from the following easy computation:

$$p^{-dg} \times p^{-2g\#\mathfrak{T}} \alpha_\pi^{2(i+\#\mathfrak{T})} \beta_\pi^{2(d-i+\#\mathfrak{T})} \times (\alpha_\pi \beta_\pi / p^g)^{-(d+2\#\mathfrak{T})} = (\alpha_\pi / \beta_\pi)^{2i-d}.$$

To prove (5.4.3), we first compute the canonical lift of the link morphism $((\eta_{\mathfrak{s},(0)}^{2d})_\sharp'', (\eta_{\mathfrak{s},(0)}^{2d})''^\sharp)$ to an endomorphism of $\mathrm{Sh}_{K_p}(G_{\mathfrak{s},\mathfrak{T}}) \times \mathrm{Sh}_{K_{E,p}}(T_{E,\mathfrak{s},\mathfrak{T}})$ appearing in Construction 2.15 Step I (and the shift in Step II is trivial in our case). This lift is clearly a composition of the Frobenius endomorphism relative to $\mathbb{F}_{p^{2g}}$, which we denote by Φ_\times^{2g} , and the action of a Hecke operator given

by a central element \mathbf{x} in $G(\mathbb{A}^\infty) \times \mathbb{A}_E^{\infty, \times}$. This central element \mathbf{x} is characterized by (and uniquely determined by) the following two conditions:

- (a) the resulting link morphism on $\mathrm{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}}) \times \mathrm{Sh}_{K_{E,p}}(T_{E, \tilde{\mathbf{S}}, \mathbf{T}})$ preserves the neutral connected component, and
- (b) under the natural projection $G(\mathbb{A}^\infty) \times \mathbb{A}_E^{\infty, \times} \rightarrow G(\mathbb{A}^\infty) \times_{\mathbb{A}_F^{\infty, \times}} \mathbb{A}_E^{\infty, \times} \cong G''(\mathbb{A}^\infty)$, \mathbf{x} is mapped to the central element $((\underline{p}_F)^g, 1)$ if \mathbf{p} is inert in E/F and to $((\underline{p}_F)^g, (\underline{\mathbf{q}})^{\Delta_{\tilde{\mathbf{S}}}})$ if \mathbf{p} splits in E/F .

We claim that $\mathbf{x} = ((\underline{p}_F)^{\#S_\infty^c + 2\#\mathbf{T}}, (\underline{p}_F)^{\#S_\infty - 2\#\mathbf{T}})$ if \mathbf{p} is inert in E/F , and $\mathbf{x} = ((\underline{p}_F)^{\#S_\infty^c + 2\#\mathbf{T}}, (\underline{p}_F)^{\#S_\infty - 2\#\mathbf{T}}(\underline{\mathbf{q}})^{\Delta_{\tilde{\mathbf{S}}}})$ if \mathbf{p} splits in E/F . Clearly, this element satisfies (b) above. To see (a), we note that the action of Φ_\times^{2g} on the geometric connected component is the image of the *arithmetic Frobenius* $\mathrm{Frob}_{p^{2g}}^{-1}$ under the Shimura reciprocity maps in Subsections 2.2 and 2.7, namely

$$\begin{cases} ((\underline{p}_F)^{-2\#S_\infty^c - 4\#\mathbf{T}}, (\underline{p}_F)^{2\#\mathbf{T} - \#S_\infty}) & \text{if } \mathbf{p} \text{ is inert in } E/F, \\ ((\underline{p}_F)^{-2\#S_\infty^c - 4\#\mathbf{T}}, (\underline{p}_F)^{2\#\mathbf{T} - \#S_\infty}(\underline{\mathbf{q}})^{-\Delta_{\tilde{\mathbf{S}}}} & \text{if } \mathbf{p} \text{ splits in } E/F. \end{cases}$$

But this element is exactly $(\nu \times \mathrm{id})(\mathbf{x}^{-1})$.

Now, taking the fiber over $\mathbf{1} \in \mathrm{Sh}_{K_{E,p}}(T_{E, \tilde{\mathbf{S}}, \mathbf{T}})$ tells us that the (canonical) link morphism $(\eta_{\mathbf{S}}^d)_{(0), \#}$ is the Frobenius endomorphism $\Phi_{G_{\mathbf{S}, \mathbf{T}}}^{2g}$ on $\mathrm{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})$ relative to $\mathbb{F}_{p^{2g}}$ composed with the Hecke operator given by multiplication by the first coordinate of \mathbf{x} , namely, $S_p^{-\#S_\infty^c - 2\#\mathbf{T}} = S_p^{-d - 2\#\mathbf{T}}$. For the action of $(\eta_{\mathbf{S}}^d)_{(0)}^*$ on $H_{\mathrm{et}}^d(\mathrm{Sh}_K(G_{\mathbf{S}, \mathbf{T}})_{\overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{S}, \mathbf{T}}^{(k, w)})$, we note that the induced action of the Frobenius endomorphism $\Phi_{G_{\mathbf{S}, \mathbf{T}}}^{2g}$ on cohomology coincides with $\mathrm{Frob}_{p^{2g}}$ (as opposed to the arithmetic Frobenius). So we have $(\eta_{\mathbf{S}}^d)_{(0)}^* = p^{-dg} \cdot \mathrm{Frob}_{p^{2g}} \circ S_p^{-d - 2\#\mathbf{T}}$, where p^{-dg} is the normalization factor in (2.25.1). This proves (2.27.1) and hence the Proposition. \square

2.28. Link morphisms II. If $\eta : \mathbf{S} \rightarrow \mathbf{S}'$ is a general link without the assumption that all curves of η are turning to the right, there exists an integer $N \geq 0$ such that the composition of the links $\xi := \eta \circ \sigma^{2gN} = \eta \circ (\eta_{\mathbf{S}}^{2d})^N = (\eta_{\mathbf{S}'}^{2d})^N \circ \eta : \mathbf{S} \rightarrow \mathbf{S}'$ satisfies the assumption, where $\eta_{\mathbf{S}}$ is the fundamental link for \mathbf{S} (2.20). Suppose that the link morphism on $\mathrm{Sh}_{K_p}(G_{\mathbf{S}}'')$ associated to ξ with indentation degree n exists. Then we put, for each $\pi \in \mathcal{A}^{(k, w)}$,

$$\begin{aligned} \eta_{(n)}^* : H_{\mathrm{et}}^d(\mathrm{Sh}_{K_p}(G_{\mathbf{S}', \mathbf{T}'}))_{\overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{S}', \mathbf{T}'}^{(k, w)} & \xrightarrow{((\eta_{\mathbf{S}}^{2d})_{(0)}^*)^{-N}} H_{\mathrm{et}}^d(\mathrm{Sh}_{K_p}(G_{\mathbf{S}', \mathbf{T}'}))_{\overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{S}', \mathbf{T}'}^{(k, w)}[\pi] \\ & \xrightarrow{\xi_{(n)}^*} H_{\mathrm{et}}^d(\mathrm{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}}))_{\overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{S}, \mathbf{T}}^{(k, w)}[\pi], \end{aligned}$$

and call it the *normalized link morphism* on the cohomology group of quaternionic Shimura varieties associated to η with indentation degree n . Here the link morphism $(\eta_{\mathbf{S}}^{2d})_{(0)}^*$ is taken to be the canonical one, so that it is invertible by Proposition 2.27. The shift of $\eta_{(n)}^*$ is defined to be the same as that of $\xi_{(n)}^*$ (as $(\eta_{\mathbf{S}}^{2d})_{(0)}^*$ has shift $\mathbf{1}$). By Lemma 2.22 on the uniqueness of link morphisms, this definition does not depend on the choice of N (but on the shift of $\xi_{(n)}^*$) and is compatible with compositions.

Lemma 2.29. (1) *For any link $\eta : \mathbf{S} \rightarrow \mathbf{S}'$, there exist an integer $N > 0$ and another link $\xi : \mathbf{S}' \rightarrow \mathbf{S}$ turning to the right such that $\xi \circ \eta : \mathbf{S} \rightarrow \mathbf{S}$ is the same as $\sigma^{2gN} : \mathbf{S} \rightarrow \mathbf{S}$.*

(2) *If $\eta : \mathbf{S} \rightarrow \mathbf{S}'$ is a link with all curves turning to the right, and the link morphism $(\eta_{(n), \#}'', \eta_{(n), \#}'')$ on $\mathrm{Sh}_{K_p}(G_{\mathbf{S}}'')$ with indentation degree n associated to η exists, then there exists $N > 0$ such that the link morphism associated to $\eta^{-1} \circ (\eta_{\mathbf{S}'}^{2d})^N : \mathbf{S}' \rightarrow \mathbf{S}$ of indentation $-n$ exists.*

(3) *Let $\eta : \mathbf{S} \rightarrow \mathbf{S}'$ be the link as in (2), and $\eta_{(n), \#} : \mathrm{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}}) \rightarrow \mathrm{Sh}_{K_p}(G_{\mathbf{S}', \mathbf{T}'})$ be the link morphism with some shift \mathbf{t} obtained by applying Construction 2.15 to $\eta_{(n)}''$. If $\eta^{-1} : \mathbf{S}' \rightarrow \mathbf{S}$*

denotes the inverse link, then the morphism

$$(\eta^{-1})_{(-n)}^* : H_{\text{et}}^d(\text{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})_{\overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{S}, \mathbf{T}}^{(k, w)}) \longrightarrow H_{\text{et}}^d(\text{Sh}_{K_p}(G_{\mathbf{S}', \mathbf{T}'})_{\overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{S}', \mathbf{T}'}^{(k, w)})$$

with some shift \mathbf{t}^{-1} is the same as the inverse of $\eta_{(n)}^*$, which has shift \mathbf{t} . Moreover, if $\eta_{(n), \#}''$ (or equivalently $\eta_{(n), \#}$) is finite flat of degree $p^{v(\eta)}$, where $v(\eta)$ denotes the total displacement of η , we have also $(\eta^{-1})_{(-n)}^* = p^{-v(\eta)/2} \text{Tr}_{\eta_{(n), \#}}$, where $\text{Tr}_{\eta_{(n), \#}}$ is the trace map on cohomology induced by the finite flat morphism $\eta_{(n), \#}$.

Proof. (1) is obvious. For (2), we may first find N so that $\xi := \eta^{-1} \circ (\eta_{\mathbf{S}'}^{2d})^N$ has all curves turning to the right. Then we consider the two morphisms

$$\begin{array}{ccc} \text{Sh}_{K_p''}(G_{\tilde{\mathbf{S}}}''') & & \text{Sh}_{K_p''}(G_{\tilde{\mathbf{S}}'}''') \\ & \searrow \eta_{(n), \#} & \swarrow (\eta_{\tilde{\mathbf{S}}'}^{2d})_{(0), \#}^N \\ & \text{Sh}_{K_p''}(G_{\tilde{\mathbf{S}}'}''') & \end{array}$$

Since the link morphism $\eta_{(n), \#}$ induces a bijection on the closed points, [He12, Proposition 4.8] implies that after possibly enlarging N , the map $(\eta_{\tilde{\mathbf{S}}'}^{2d})_{(0), \#}^N$ factors through $\eta_{(n), \#}$, as $\eta_{(n), \#} \circ \xi_{\#}$. It is easy to see that $\xi_{\#}$ gives the required link morphism.

The first part of (3) follows from the uniqueness of link morphism (Lemma 2.22). For the second part of (3), note that the composed morphism

$$H_{\text{et}}^d(\text{Sh}_{K_p}(G_{\mathbf{S}', \mathbf{T}'})_{\overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{S}', \mathbf{T}'}^{(k, w)}) \xrightarrow{p^{v(\eta)/2} \eta_{(n)}^*} H_{\text{et}}^d(\text{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})_{\overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{S}, \mathbf{T}}^{(k, w)}) \xrightarrow{\text{Tr}_{\eta_{(n), \#}}} H_{\text{et}}^d(\text{Sh}_{K_p}(G_{\mathbf{S}', \mathbf{T}'})_{\overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{S}', \mathbf{T}'}^{(k, w)}),$$

is nothing but the multiplication by $p^{v(\eta)}$, according to our normalization of $\eta_{(n), \#}^*$ (2.25.1). It follows immediately that $(\eta^{-1})_{(-n)}^* = p^{-v(\eta)/2} \text{Tr}_{\eta_{(n), \#}}$. \square

2.30. Goren–Oort divisors. We recall the definition of the Goren–Oort stratification from [TX13⁺a, Section 4]. We will make essential use of the case of divisors. Let $\text{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})$ be the special fiber of a quaternionic Shimura variety of type considered in Subsection 2.2. We fix throughout this paper a choice of lifting $\tilde{\mathbf{S}}_{\infty}$ of \mathbf{S}_{∞} , and let $\text{Sh}_{K_p''}(G_{\tilde{\mathbf{S}}}''')$ be the associated unitary Shimura variety.

In [TX13⁺a, Definition 4.6 and §4.9], we defined, for each $\tau \in \mathbf{S}_{\infty}^c$, the *Goren–Oort divisor* $\text{Sh}_{K_p''}(G_{\tilde{\mathbf{S}}}''')_{\tau}$ of $\text{Sh}_{K_p''}(G_{\tilde{\mathbf{S}}}''')$ at τ as the vanishing locus of the τ -th partial Hasse invariant of the versal family $\mathbf{A}_{\tilde{\mathbf{S}}}''$. Each $\text{Sh}_{K_p''}(G_{\tilde{\mathbf{S}}}''')_{\tau}$ is projective and smooth by [TX13⁺a, Proposition 4.7]. Transferring these structures to the quaternionic Shimura varieties using Proposition 2.10, one gets a Goren–Oort divisor $\text{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})_{\tau}$ on $\text{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})$ for each $\tau \in \mathbf{S}_{\infty}^c$. When $\mathbf{T} = \emptyset$, this is done in [TX13⁺a, 4.9], and the general case is the same.

For a subset $J \subset \mathbf{S}_{\infty}^c$, we put $\text{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})_J = \cap_{\tau \in J} \text{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})_{\tau}$ and $\text{Sh}_{K_p''}(G_{\tilde{\mathbf{S}}}''')_J = \cap_{\tau \in J} \text{Sh}_{K_p''}(G_{\tilde{\mathbf{S}}}''')_{\tau}$. The closed subvarieties $\text{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})_J$ (resp. $\text{Sh}_{K_p''}(G_{\tilde{\mathbf{S}}}''')_J$) with J running through the subsets of \mathbf{S}_{∞}^c form the Goren–Oort stratification of $\text{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})$ (resp. $\text{Sh}_{K_p''}(G_{\tilde{\mathbf{S}}}''')$).

The main results of [TX13⁺a] give an explicit description of all closed Goren–Oort strata $\text{Sh}_{K_p''}(G_{\tilde{\mathbf{S}}}''')_J$ (resp. $\text{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})_J$) as iterated \mathbb{P}^1 -bundles over another unitary (resp. quaternionic) Shimura variety of the same type. We list results from [TX13⁺a] that we will make use in this paper. (One more result will be used later in proving Lemma 5.14.)

Proposition 2.31. *Let $\tau \in \mathbf{S}_{\infty}^c$. Assume that $\tau^- = \sigma^{-n_{\tau}} \tau$ is different from τ (See 2.19 for the notation). We put $\mathbf{S}_{\tau} = \mathbf{S} \cup \{\tau, \tau^-\}$ and $\mathbf{T}_{\tau} = \mathbf{T} \cup \{\tau\}$. Let $\tilde{\mathbf{S}}_{\tau, \infty}$ be the lifting of $\mathbf{S}_{\tau, \infty}$ derived from $\tilde{\mathbf{S}}_{\infty}$ according to the rule of [TX13⁺a, 5.3], and put $\tilde{\mathbf{S}}_{\tau} = (\mathbf{S}_{\tau}, \tilde{\mathbf{S}}_{\tau, \infty})$. In particular, $\Delta_{\tilde{\mathbf{S}}_{\tau}} = \Delta_{\tilde{\mathbf{S}}}$ when \mathfrak{p} splits in E/F .*

(1) *There exists a \mathbb{P}^1 -bundle fibration*

$$\pi''_\tau : \mathrm{Sh}_{K_p''}(G''_{\tilde{\mathbf{S}}})_\tau \rightarrow \mathrm{Sh}_{K_p''}(G''_{\tilde{\mathbf{S}}_\tau})_\tau$$

equivariant for the action of $\mathcal{G}''_{\tilde{\mathbf{S}},p} = \mathcal{G}''_{\tilde{\mathbf{S}}_\tau,p}$, and a p -quasi-isogeny of abelian schemes on $\mathrm{Sh}_{K_p''}(G''_{\tilde{\mathbf{S}}})_\tau$

$$\Phi_{\pi''_\tau} : \mathbf{A}''_{\tilde{\mathbf{S}}} \rightarrow \pi''^{**}(\mathbf{A}''_{\tilde{\mathbf{S}}_\tau}).$$

By Construction 2.15, this gives rise to a \mathbb{P}^1 -bundle fibration

$$\pi_\tau : \mathrm{Sh}_{K_p}(G_{\mathbf{S},\mathbf{T}})_\tau \longrightarrow \mathrm{Sh}_{K_p}(G_{\mathbf{S}_\tau,\mathbf{T}_\tau})$$

with some shift $\mathbf{t}_\tau = \mathbf{t}_\tau(\mathbf{S}, \mathbf{T}) \in E^{\times, \mathrm{cl}} \backslash \mathbb{A}_E^{\infty, \times} / \mathcal{O}_{E_p}^\times$, which is unique up to $F^{\times, \mathrm{cl}} \backslash \mathbb{A}_F^{\infty, \times} / \mathcal{O}_{F_p}^\times$ and compatible with the Hecke action of $G(\mathbb{A}^{\infty, p})$, together with an isomorphism of étale sheaves for each given regular multiweight (k, w)

$$\pi_\tau^\# : \mathcal{L}_{\mathbf{S},\mathbf{T}}^{(k,w)}|_{\mathrm{Sh}_{K_p}(G_{\mathbf{S},\mathbf{T}})_\tau} \xrightarrow{\sim} \pi_\tau^*(\mathcal{L}_{\mathbf{S}_\tau,\mathbf{T}_\tau}^{(k,w)}).$$

The morphisms π_τ and $\pi_\tau^\#$ are unique up to a central Hecke action by an element of $F^{\times, \mathrm{cl}} \backslash \mathbb{A}_F^{\infty, \times} / \mathcal{O}_{F_p}^\times$.

(2) *Let $\mathcal{O}(1)$ be the tautological quotient line bundle on $\mathrm{Sh}_{K_p}(G_{\mathbf{S},\mathbf{T}})_\tau$ for the \mathbb{P}^1 -bundle given by π_τ . If $\tau^- = \sigma^{-n_\tau}\tau$ is different from τ , then the normal bundle of the closed immersion $\mathrm{Sh}_{K_p}(G_{\mathbf{S},\mathbf{T}})_\tau \hookrightarrow \mathrm{Sh}_{K_p}(G_{\mathbf{S},\mathbf{T}})$ is, up to tensoring a line bundle which is torsion in the Picard group of $\mathrm{Sh}_{K_p}(G_{\mathbf{S},\mathbf{T}})_\tau$, the same as $\mathcal{O}(-2p^{n_\tau}) = \mathcal{O}(1)^{\otimes(-2p^{n_\tau})}$.*

Proof. In statement (1), the existence of π''_τ is a special case of [TX13⁺a, Corollary 5.9]. Roughly speaking, this \mathbb{P}^1 -bundle π''_τ parametrizes the lines (the Hodge filtration) in the reduced $\tilde{\tau}^- = \sigma^{-n_\tau}\tilde{\tau}$ -component of the relative de Rham homology of the versal family $\mathbf{A}''_{\tilde{\mathbf{S}}_\tau}$ on $\mathrm{Sh}_{K_p''}(G''_{\tilde{\mathbf{S}}_\tau})_\tau$. It is straightforward to check that the condition (2.15.1) is satisfied for the pairs $(\tilde{\mathbf{S}}, \mathbf{T})$ and $(\tilde{\mathbf{S}}_\tau, \mathbf{T}_\tau)$. We apply Construction 2.15 to deduce the existence of $(\pi_\tau, \pi_\tau^\#)$ from that of $(\pi''_\tau, \Phi_{\pi''_\tau})$.

Statement (2) follows from [TX13⁺a, Proposition 6.4], when noting that the quaternionic Shimura varieties and the unitary Shimura varieties have isomorphic geometric connected components. \square

Proposition 2.26(1) implies that we have a morphism

$$\pi_\tau^* : H_{\mathrm{et}}^*(\mathrm{Sh}_K(G_{\mathbf{S}_\tau,\mathbf{T}_\tau})_{\overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{S}_\tau,\mathbf{T}_\tau}^{(k,w)}) \longrightarrow H_{\mathrm{et}}^*(\mathrm{Sh}_K(G_{\mathbf{S},\mathbf{T}})_{\tau,\overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{S},\mathbf{T}}^{(k,w)})$$

equivariant under the actions of the Hecke algebra $\mathbb{T}(K^p) = \overline{\mathbb{Q}}_\ell[K^p \backslash G(\mathbb{A}^{\infty, p})/K^p]$. It is canonical up to the action of the central Hecke character, which comes from the ambiguity of choosing the shift in Construction 2.15.

Theorem 2.32.

(1) *Let $\tau_1, \tau_2 \in \mathcal{S}_\infty^c$ be two places such that $\tau_1, \tau_2, \tau_1^-, \tau_2^-$ are distinct. We have a Cartesian diagram*

$$\begin{array}{ccc} \mathrm{Sh}_{K_p}(G_{\mathbf{S},\mathbf{T}})_{\{\tau_1, \tau_2\}} & \xrightarrow{\pi_{\tau_1}} & \mathrm{Sh}_{K_p}(G_{\mathbf{S}_{\tau_1}, \mathbf{T}_{\tau_1}})_{\tau_2} \\ \downarrow \pi_{\tau_2} & & \downarrow \pi_{\tau_2} \\ \mathrm{Sh}_{K_p}(G_{\mathbf{S}_{\tau_2}, \mathbf{T}_{\tau_2}})_{\tau_1} & \xrightarrow{\pi_{\tau_1}} & \mathrm{Sh}_{K_p}(G_{\mathbf{S}_{\tau_1, \tau_2}, \mathbf{T}_{\tau_1, \tau_2}}), \end{array}$$

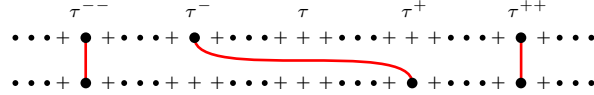
If we use the notation of shifts of these π_{τ_i} as in Proposition 2.31(1), then we have an equality

$$\mathbf{t}_{\tau_1}(\mathbf{S}, \mathbf{T}) \mathbf{t}_{\tau_2}(\mathbf{S}_{\tau_1}, \mathbf{T}_{\tau_2}) = \mathbf{t}_{\tau_2}(\mathbf{S}, \mathbf{T}) \mathbf{t}_{\tau_1}(\mathbf{S}_{\tau_2}, \mathbf{T}_{\tau_2}).$$

Moreover, we have a commutative diagram of induced morphisms on the cohomology groups:

$$\begin{array}{ccc} H_{\text{et}}^*(\text{Sh}_{K_p}(G_{\mathbf{S}_{\tau_1} \cup \mathbf{S}_{\tau_2}, \mathbf{T}_{\tau_1} \cup \mathbf{T}_{\tau_2}})_{\overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{S}_{\tau_1} \cup \mathbf{S}_{\tau_2}, \mathbf{T}_{\tau_1} \cup \mathbf{T}_{\tau_2}}^{(k,w)}) & \xrightarrow{\pi_{\tau_2}^*} & H_{\text{et}}^*(\text{Sh}_{K_p}(G_{\mathbf{S}_{\tau_1}, \mathbf{T}_{\tau_1}})_{\tau_2, \overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{S}_{\tau_1}, \mathbf{T}_{\tau_1}}^{(k,w)}) \\ \downarrow \pi_{\tau_1}^* & & \downarrow \pi_{\tau_1}^* \\ H_{\text{et}}^*(\text{Sh}_{K_p}(G_{\mathbf{S}_{\tau_2}, \mathbf{T}_{\tau_2}})_{\tau_1, \overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{S}_{\tau_2}, \mathbf{T}_{\tau_2}}^{(k,w)}) & \xrightarrow{\pi_{\tau_2}^*} & H_{\text{et}}^*(\text{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})_{\{\tau_1, \tau_2\}, \overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{S}, \mathbf{T}}^{(k,w)}). \end{array}$$

- (2) Let $\tau \in \mathbf{S}_{\infty}^c$ be a place such that τ, τ^+, τ^- are distinct. Put $n = n_{\tau^+} - n_{\tau}$ if \mathfrak{p} splits in E/F and $n = 0$ if \mathfrak{p} is inert in E/F . Let $\eta : \mathbf{S}_{\tau^+} = \mathbf{S} \cup \{\tau^+, \tau\} \rightarrow \mathbf{S}_{\tau} = \mathbf{S} \cup \{\tau, \tau^-\}$ be the link given by straight lines except sending τ^- to τ^+ over τ :



Let $\eta_{(n), \sharp}$ be the morphism defined by the following commutative diagram

$$(2.32.1) \quad \begin{array}{ccccc} \text{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})_{\tau^+} & \xleftarrow{\quad} & \text{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})_{\{\tau^+, \tau\}} & \xrightarrow{\quad} & \text{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})_{\tau} \\ \pi_{\tau^+} \downarrow & \swarrow \cong & & & \downarrow \pi_{\tau} \\ \text{Sh}_{K_p}(G_{\mathbf{S}_{\tau^+}, \mathbf{T}_{\tau^+}}) & \xrightarrow{\eta_{(n), \sharp}} & & & \text{Sh}_{K_p}(G_{\mathbf{S}_{\tau}, \mathbf{T}_{\tau}}) \end{array}$$

Then the following statements hold:

- (a) The map $\eta_{(n), \sharp}$ is the morphism obtained by applying Construction 2.15 to a link morphism on $\text{Sh}_{K_p}''(G_{\mathbf{S}_{\tau^+}}'')$ with indentation degree n .
- (b) If $\mathbf{t}_{\tau} \in E^{\times, \text{cl}} \backslash \mathbb{A}_E^{\infty, \times} / \mathcal{O}_{E, \mathfrak{p}}^{\times}$ for $\tau = \tau, \tau^+$ denotes the shift of the correspondence

$$\text{Sh}_{K_p}(G_{\mathbf{S}_{\tau}, \mathbf{T}_{\tau}}) \xleftarrow{\pi_{\tau}^*} \text{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})_{\tau} \hookrightarrow \text{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}}),$$

then $\eta_{(n), \sharp}$ has shift $\mathbf{t}_{\tau^+} \mathbf{t}_{\tau}^{-1}$.

- (c) The morphism $\eta_{(n), \sharp}$ is finite flat of degree $p^{v(\eta)}$.
- (d) The p -quasi-isogenies of the versal families of abelian varieties on $\text{Sh}_{K_p}''(G_{\mathbf{S}_{\tau^+}}'')$ given by

$$\pi_{\tau^+}''^*(\mathbf{A}_{\mathbf{S}_{\tau^+}}'')|_{\text{Sh}_{K_p}''(G_{\mathbf{S}}'')_{\{\tau^+, \tau\}}} \xleftarrow{\Phi_{\pi_{\tau^+}''}} \mathbf{A}_{\mathbf{S}}''|_{\text{Sh}_{K_p}''(G_{\mathbf{S}}'')_{\{\tau^+, \tau\}}} \xrightarrow{\Phi_{\pi_{\tau}''}} \pi_{\tau}''^*(\mathbf{A}_{\mathbf{S}_{\tau}}'')|_{\text{Sh}_{K_p}''(G_{\mathbf{S}}'')_{\{\tau^+, \tau\}}}$$

induces a link morphism on the sheaves $\eta_{(n), \sharp}^{\#} : \mathcal{L}_{\mathbf{S}_{\tau^+}, \mathbf{T}_{\tau^+}}^{(k,w)} \rightarrow \eta_{(n), \sharp}^*(\mathcal{L}_{\mathbf{S}_{\tau}, \mathbf{T}_{\tau}}^{(k,w)})$. Then the induced normalized link morphism $\eta_{(n)}^*$ on the cohomology groups constructed as in 2.25 fits into the following commutative diagram:

$$(2.32.2) \quad \begin{array}{ccccc} H_{\text{et}}^*(\text{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})_{\tau^+, \overline{\mathbb{F}}_p}) & \longrightarrow & H_{\text{et}}^*(\text{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})_{\{\tau^+, \tau\}, \overline{\mathbb{F}}_p}) & \longleftarrow & H_{\text{et}}^*(\text{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})_{\tau, \overline{\mathbb{F}}_p}) \\ \pi_{\tau^+}^* \uparrow & \nearrow \cong & & & \uparrow \pi_{\tau}^* \\ H_{\text{et}}^*(\text{Sh}_{K_p}(G_{\mathbf{S}_{\tau^+}, \mathbf{T}_{\tau^+}})_{\overline{\mathbb{F}}_p}) & \xleftarrow{p^{(n_{\tau} + n_{\tau^+})/2} \eta_{(n)}^*} & & & H_{\text{et}}^*(\text{Sh}_{K_p}(G_{\mathbf{S}_{\tau}, \mathbf{T}_{\tau}})_{\overline{\mathbb{F}}_p}) \end{array}$$

where the upper horizontal arrows are natural restriction maps. Here, for simplification, we have suppressed the sheaves from the notation. For instance, $H_{\text{et}}^*(\text{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})_{\tau^+, \overline{\mathbb{F}}_p})$ should be understood as $H_{\text{et}}^*(\text{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})_{\tau^+, \overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{S}, \mathbf{T}}^{(k,w)}|_{\text{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})_{\tau^+}})$.

- (3) Assume that $\mathbf{S}_\infty^c = \{\tau, \tau^-\}$ (and hence \mathfrak{p} splits in E/F). Then $\mathrm{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})_{\{\tau, \tau^-\}}$ is isomorphic to the special fiber of the zero-dimensional Shimura variety $\mathrm{Sh}_{\mathrm{Iw}_p}(G_{\mathbf{S}_\tau, \mathbf{T}_\tau})$ of Iwahori level at \mathfrak{p} . Let $\eta : \mathbf{S}_{\tau^-} \rightarrow \mathbf{S}_\tau$ denote the link map (with no curve). Then the link morphism $\eta_{(n_{\tau^-})}^\# : \mathrm{Sh}_{K_p}(G_{\mathbf{S}_{\tau^-}, \mathbf{T}_{\tau^-}}) \xrightarrow{\sim} \mathrm{Sh}_{K_p}(G_{\mathbf{S}_\tau, \mathbf{T}_\tau})$ of indentation degree $2n_{\tau^-}$ associated to η exists, and the following diagram

$$\begin{array}{ccc} & \mathrm{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})_{\{\tau, \tau^-\}} & \\ \pi_\tau \swarrow & & \searrow \pi_{\tau^-} \\ \mathrm{Sh}_{K_p}(G_{\mathbf{S}_\tau, \mathbf{T}_\tau}) & & \mathrm{Sh}_{K_p}(G_{\mathbf{S}_{\tau^-}, \mathbf{T}_{\tau^-}}) \xrightarrow[\eta_{(n_{\tau^-})}]{\cong} \mathrm{Sh}_{K_p}(G_{\mathbf{S}_\tau, \mathbf{T}_\tau}). \end{array}$$

is (the base change to \mathbb{F}_{p^g} of) the Hecke correspondence $T_{\mathfrak{p}}$ on $\mathrm{Sh}_{K_p}(G_{\mathbf{S}_\tau, \mathbf{T}_\tau})$. If $\mathbf{t}_? \in E^{\times, \mathrm{cl}} \backslash \mathbb{A}_E^{\infty, \times} / \mathcal{O}_{E_{\mathfrak{p}}}^\times$ for $? = \tau, \tau^+$ denotes the shift of the correspondence

$$\mathrm{Sh}_{K_p}(G_{\mathbf{S}_?, \mathbf{T}_?}) \xleftarrow{\pi_?} \mathrm{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}}) \hookrightarrow \mathrm{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}}),$$

then $\eta_{(n_{\tau^-})}^\#$ has shift $\varpi_{\bar{\mathfrak{q}}} \mathbf{t}_\tau^{-1} \mathbf{t}_{\tau^-}$. Moreover, the map induced by the diagram above on cohomology groups

$$H_{\mathrm{et}}^0(\mathrm{Sh}_K(G_{\mathbf{S}_\tau, \mathbf{T}_\tau})_{\overline{\mathbb{F}}_p}) \xrightarrow{(\eta_{(n_{\tau^-})} \circ \pi_{\tau^-})^*} H_{\mathrm{et}}^0(\mathrm{Sh}_K(G_{\mathbf{S}, \mathbf{T}})_{\{\tau, \tau^-\}, \overline{\mathbb{F}}_p}) \xrightarrow{\mathrm{Tr}_{\pi_\tau}} H_{\mathrm{et}}^0(\mathrm{Sh}_K(G_{\mathbf{S}_\tau, \mathbf{T}_\tau})_{\overline{\mathbb{F}}_p}).$$

is the usual Hecke action $T_{\mathfrak{p}}$. Here, as in (2), we have suppressed the sheaves from the notation.

Proof. The analogs of (1), (2)(a) and (2)(c) for unitary Shimura varieties were proved in [TX13⁺a, Proposition 7.12 and Theorem 7.16]. The statements here follow from Construction 2.15.

Statement (2)(b) regarding shifts follows directly from Remark 2.16. Statement (2)(d) is direct by the construction of $\eta_{(n)}^\#$ and $\eta_{(n)}^*$.

For (3), the analogous statement for unitary Shimura varieties $\mathrm{Sh}_{K''}(G_{\mathbf{S}'})$ (with $T_{\mathfrak{p}}$ replaced by $T_{\mathfrak{q}}$) was proved in [TX13⁺a, Theorem 7.16(2)]. One deduces (3) using the construction in Subsection 2.17, and computes the shifts by Remark 2.16. \square

3. GOREN–OORT CYCLES

In this section, we investigate certain generalization of Goren–Oort strata, called Goren–Oort cycles. They are parametrized by certain combinatorial data, called *periodic semi-meanders*. We will show later that the intersection matrix of the Goren–Oort cycles turns out to be closely related to the Gram matrix associated to these periodic semi-meanders (which explains our choice of the combinatorial model).

3.1. Periodic semi-meanders. The combinatorial construction that we will use later is related to the so-called link representations of periodic Temperley–Lieb algebras, which appear naturally in the study of mathematical physics; see for example [Di90, GL98, MS13]. We will simply state here the main result with minimal input, and refer to [MS13] for a detailed discussion of the mathematical physics background and the proofs.

We slightly modify the usual definition of periodic semi-meanders to adapt to our situation. Recall that F is a totally real field of degree g and \mathbf{S}, \mathbf{T} are introduced as in Subsection 2.2, and $d = \#\mathbf{S}_\infty^c$. We consider the band associated to \mathbf{S} defined as in Subsection 2.18, and recall that the band is placed on a cylinder, but we often draw it over the 2-dimensional xy -plane with x -coordinate taken modulo g .

A *periodic semi-meander* for \mathbf{S} is a collection of curves (called *arcs*) that link two nodes of the band for \mathbf{S} , and straight lines (called *semi-lines*) that links a node to the infinity ($+\infty$ in the y -direction) subject to the following conditions:

- All the arcs and semi-lines lie on the cylinder above the band (that is to have positive y -coordinate in the 2-dimensional picture).
- Each node of the band for \mathbf{S} is exactly one end point of an arc or a semi-line.
- There are no intersection points among these arcs and semi-lines.

The number of arcs is denoted by r (so $r \leq d/2$), and the number of semi-lines $d - 2r$ is called the *defect* of the periodic semi-meander. Two periodic semi-meanders are considered as the same if they can be continuously deformed into each other while keeping the above three properties in the process. We use $\mathfrak{B}_{\mathbf{S}}^r$ denote the set of semi-meanders for \mathbf{S} with r arcs (up to the deformations). For example, if F has degree 7 over \mathbb{Q} , $r = 2$, and $\mathbf{S} = \{\infty_1, \infty_4\}$, we have

$$(3.1.1) \quad \mathfrak{B}_{\mathbf{S}}^2 = \left\{ \begin{array}{c} \downarrow + \bullet + \bullet + \bullet + \bullet + \bullet + \bullet + \downarrow, \downarrow + \bullet + \bullet + \bullet + \bullet + \bullet + \downarrow, \downarrow + \bullet + \bullet + \bullet + \bullet + \bullet + \downarrow, \downarrow + \bullet + \bullet + \bullet + \bullet + \bullet + \downarrow, \\ \downarrow + \bullet + \bullet + \bullet + \bullet + \bullet + \downarrow, \downarrow + \bullet + \bullet + \bullet + \bullet + \bullet + \downarrow, \downarrow + \bullet + \bullet + \bullet + \bullet + \bullet + \downarrow, \downarrow + \bullet + \bullet + \bullet + \bullet + \bullet + \downarrow, \\ \downarrow + \bullet + \bullet + \bullet + \bullet + \bullet + \downarrow, \downarrow + \bullet + \bullet + \bullet + \bullet + \bullet + \downarrow, \downarrow + \bullet + \bullet + \bullet + \bullet + \bullet + \downarrow, \downarrow + \bullet + \bullet + \bullet + \bullet + \bullet + \downarrow \end{array} \right\}.$$

When drawing in the xy -plane, points are placed on the x -axis at points of coordinates $(0, 0), \dots, (g-1, 0)$ and the diagram for a periodic semi-meander is taken to be periodic in the x -direction of period g . The curves connecting the points can connect across the imaginary boundary lines at $x = -1/2$ and $x = g - 1/2$ (which are identified). See for example (3.1.1). An elementary calculation shows that $\#\mathfrak{B}_{\mathbf{S}}^r = \binom{d}{r}$.

A *standard presentation* of a semi-meander is where all the arcs are monotonic in the x -direction, namely, it does not twist back and forth. Using the xy -plane picture, we define the *left* and *right end-nodes* of an arc, as follows:

- when the arc appears as one arc in the standard presentation, its left (resp. right) end-node is the left (resp. right) endpoint of the arc;
- when the arc appears in two parts linked through the imaginary boundary lines at $x = -1/2$ and $x = g - 1/2$, its left (resp. right) end-node is the right (resp. left) endpoint of the arc

For $\mathbf{a} \in \mathfrak{B}_{\mathbf{S}}^r$, we use $\ell(\mathbf{a})$ to denote the *total span* of \mathbf{a} , that is the sum of the span of all curves over the band, where the span takes into account of the periodicity at the imaginary boundary. For example, the last element of $\mathfrak{B}_{\mathbf{S}}^2$ in (3.1.1) has two arcs with spans 1 and 5, respectively, and hence its total span is 6. The second element of $\mathfrak{B}_{\mathbf{S}}^2$ in (3.1.1) has two arcs with spans 1 and 2, respectively, and hence its total span is 3.

3.2. Gram matrix. For $\mathbf{a}, \mathbf{b} \in \mathfrak{B}_{\mathbf{S}}^r$, we consider the drawing $D(\mathbf{a}, \mathbf{b})$ obtained by taking mirror image of \mathbf{b} reflected about the x -axis and then identifying the d nodes of \mathbf{b} with those of \mathbf{a} according to their labellings.

- We say a loop (namely, a closed curve) in $D(\mathbf{a}, \mathbf{b})$ is *contractible* if it can be continuously contracted to a point on the cylinder (ignoring all other curves and lines on the picture). We write $m_0(\mathbf{a}, \mathbf{b})$ for the number of contractible loops in $D(\mathbf{a}, \mathbf{b})$.
- We say a loop in $D(\mathbf{a}, \mathbf{b})$ is *non-contractible* if, ignoring other curves and lines on the picture, it can be continuously deformed into a loop wrapped around the cylinder. (Since all loops do not intersect itself, the loop can only wrap the cylinder for one round.) We write $m_T(\mathbf{a}, \mathbf{b})$ for the number of non-contractible loops in $D(\mathbf{a}, \mathbf{b})$. This can only happen if $r = d/2$.
- We use $\mathbf{S}_{\mathbf{a}}$ to denote the union of \mathbf{S} with the nodes that are connected to an arc of \mathbf{a} . So the band of $\mathbf{S}_{\mathbf{a}}$ maybe obtained from the band of \mathbf{a} by replacing the end-nodes of arcs in \mathbf{a} by plus signs. We define $\mathbf{S}_{\mathbf{b}}$ similarly.
- Assume that $r < d/2$, that no semi-line of \mathbf{a} is connected to another semi-line of \mathbf{a} in $D(\mathbf{a}, \mathbf{b})$, and that the same is true for \mathbf{b} . We define the *reduction* of $D(\mathbf{a}, \mathbf{b})$ to be a link $\eta_{\mathbf{S}_{\mathbf{a}}, \mathbf{S}_{\mathbf{b}}}$ from the band of $\mathbf{S}_{\mathbf{a}}$ to the band of $\mathbf{S}_{\mathbf{b}}$ such that each node $\tau_{\mathbf{a}}$ of $\mathbf{S}_{\mathbf{a}}$ (corresponding to an element of $\mathbf{S}_{\mathbf{a}, \infty}^c$) is linked to a node $\tau_{\mathbf{b}}$ of $\mathbf{S}_{\mathbf{b}}$ in the same way as the semi-line at $\tau_{\mathbf{a}}$ is linked to the semi-line at $\tau_{\mathbf{b}}$ in $D(\mathbf{a}, \mathbf{b})$. In practice, this amounts to removing all the (contractible) loops

- When $r = \frac{d}{2}$, $S_a = S_b$ contains all the archimedean places. For consistency, we write η_{S_a, S_b} for the trivial link from the band of S_a to the band of S_b (as there are no nodes on the bands).

$$\langle \cdot | \cdot \rangle_{\mathbf{S}} : \mathfrak{B}_{\mathbf{S}}^r \times \mathfrak{B}_{\mathbf{S}}^r \longrightarrow \begin{cases} \overline{\mathbb{Q}}_{\ell}(v) & \text{if } r < d/2, \\ \overline{\mathbb{Q}}_{\ell}[T] & \text{if } r = d/2. \end{cases}$$

$$\langle \mathbf{a} | \mathbf{b} \rangle_{\mathbf{S}} = \begin{cases} 0 & \text{if in the diagram } D(\mathbf{a}, \mathbf{b}), \text{ two semi-lines} \\ & \text{of } \mathbf{a}(\text{or of } \mathbf{b}) \text{ are connected,} \\ (-2)^{m_0(\mathbf{a}, \mathbf{b})} v^{m_v(\mathbf{a}, \mathbf{b})} & \text{otherwise if } r < d/2, \text{ and} \\ (-2)^{m_0(\mathbf{a}, \mathbf{b})} T^{m_T(\mathbf{a}, \mathbf{b})} & \text{otherwise if } r = d/2. \end{cases}$$

Example 3.3. The following examples are copied from [MS13].

(2) $\alpha = \bullet \bullet \bullet \bullet \bullet \bullet$, $\beta = \bullet \bullet \bullet \bullet \bullet \bullet$, $D(\alpha, \beta) =$ $, \text{ and } \langle \alpha | \beta \rangle_s = 0.$

(3) $\mathfrak{a} = \text{diagram 1}$, $\mathfrak{b} = \text{diagram 2}$, and $D(\mathfrak{a}, \mathfrak{b}) = \text{diagram 3}$, and $\langle \mathfrak{a} | \mathfrak{b} \rangle_{\mathcal{S}} = (-2)^3 T^2$.

The following theorem is essentially the main theorem of [MS13] (which seems to have been known by [GL98] using a different argument).

Theorem 3.5. Put $t_{d,r} = \sum_{i=0}^{r-1} \binom{d}{i}$. Let $\mathfrak{G}_{\mathbf{S}}^r$ denote the Gram matrix $(\langle \mathbf{a} | \mathbf{b} \rangle)_{\mathbf{a}, \mathbf{b} \in \mathfrak{B}_{\mathbf{S}}^r}$. Then its determinant is given as follows.

- (1) When d is even, $\det \mathfrak{G}_S^{d/2} = \pm(T^2 - 4)^{t_{d,d/2}}$.
- (2) For $r < d/2$, $\det \mathfrak{G}_S^r = \pm(v^g - v^{-g})^{2t_{d,r}}$.

Proof. When $S = \emptyset$ (so $d = g$), this is a special case of [MS13, Theorem 4.1]. Indeed, the parameter α in *loc. cit.* is T in our notation, and since the β in *loc. cit.* is -2 , the C_k in *loc. cit.* are equal to ± 1 for all k . One easily simplifies their formula to the one stated in this theorem.

The general case requires little modification, but the method of the proof may be viewed as a toy model for the proof of Theorem 4.4 later. When $r = \frac{d}{2}$, we just simply ignore all points corresponding to S_∞ . This verifies (1). So we assume $r < \frac{d}{2}$ from now on to prove (2). We use $\langle \mathbf{a} | \mathbf{b} \rangle_d$ to denote the pairing computed by removing all points from S_∞ (and shrink the cylinder accordingly) and hence with displacements computed with respect to only the d nodes. Let \mathfrak{G}_d^r denote the corresponding matrix. We need to compare $\det \mathfrak{G}_d^r$ with $\det \mathfrak{G}_S^r$, by showing that $\det \mathfrak{G}_S^r$ can be obtained by replacing all v^d in the expression of $\det \mathfrak{G}_d^r$ by v^g .

By the definition of determinant, $\det \mathfrak{G}_S^r$ is the sum over all permutations σ of the set \mathfrak{B}_S^r , of the product of the signature of σ , and, for every cycle $(\mathbf{a}_1 \dots \mathbf{a}_t)$ of the permutation σ , the product

$$(3.5.1) \quad \langle \mathbf{a}_1 | \mathbf{a}_2 \rangle_S \cdot \langle \mathbf{a}_2 | \mathbf{a}_3 \rangle_S \cdots \langle \mathbf{a}_t | \mathbf{a}_1 \rangle_S.$$

The same applies to $\det \mathfrak{G}_d^r$ except that the product (3.5.1) are taken for the pairing $\langle \cdot | \cdot \rangle_d$. The product (3.5.1), if not zero, is equal to $(-2)^{m_0} v^{m_v}$, where $m_0 = m_0(\mathbf{a}_1, \mathbf{a}_2) + \cdots + m_0(\mathbf{a}_t, \mathbf{a}_1)$ is the sum of total number of contractible loops in the diagrams $D(\mathbf{a}_1, \mathbf{a}_2), D(\mathbf{a}_2, \mathbf{a}_3), \dots, D(\mathbf{a}_t, \mathbf{a}_1)$, and $m_v = m_v(\mathbf{a}_1, \mathbf{a}_2) + \cdots + m_v(\mathbf{a}_t, \mathbf{a}_1)$ is equal to the total displacement of the composition of the link

$$(3.5.2) \quad \eta_{S_{\mathbf{a}_t}, S_{\mathbf{a}_1}} \circ \cdots \circ \eta_{S_{\mathbf{a}_2}, S_{\mathbf{a}_3}} \circ \eta_{S_{\mathbf{a}_1}, S_{\mathbf{a}_2}},$$

by the additivity of total displacements as remarked in Subsection 2.18. Note that (3.5.2) is in fact a link from $S_{\mathbf{a}_1}$ to itself. So it must be an integer power n of the fundamental link $\eta_{S_{\mathbf{a}_1}}$ defined in Subsection 2.18. In particular, we have $m_v = ng$. Making the same observation for computing the standard Gram determinant $\det \mathfrak{G}_d^r$, the product (3.5.1) with $\langle \cdot | \cdot \rangle_d$ is instead equal to $(-2)^{m_0} v^{m'_v}$ with the same m_0 as above, and m'_v is the total displacement of (3.5.2) with all points corresponding to S_∞ removed. By the same discussion above, we have $m'_v = nd$ with the same n above.

In conclusion, each term of $\det \mathfrak{G}_S^r$ can be obtained from the corresponding term of $\det \mathfrak{G}_d^r$ via replacing v^d by v^g . Therefore, $\det \mathfrak{G}_S^r = \pm(v^g - v^{-g})^{2t_{d,r}}$. \square

Notation 3.6. Using the illustration of periodic semi-meanders as in (3.1.1), we say an arc δ *lies over* another arc δ' if the contractible closed loop in the picture given by adjoining δ with the equator contains δ' inside. For example, in the list of \mathfrak{B}_S^2 in (3.1.1), the last five periodic semi-meanders each has an arc lying over another.

In a periodic semi-meander for S , a *basic arc* is an arc δ which satisfies the following equivalent conditions

- in the 2-dimensional picture, δ does not lie over any other arcs,
- in the 2-dimensional picture, the only points below δ are plus signs, or
- δ is an arc which links some τ to τ^- (See 2.19 for the notation).

For example, in the list of \mathfrak{B}_S^2 in (3.1.1), the five periodic semi-meanders in the first row each has two basic arcs, and the five periodic semi-meanders in the second row each has one basic arc.

It is clear that all periodic semi-meanders have a basic arc except the one with only semi-lines. Given a periodic semi-meander $\mathbf{a} \in \mathfrak{B}_S^r$ for S with a basic arc δ linking two nodes $\tau, \tau^- \in S_\infty^c$, we can delete the arc and replace its end-nodes by $+$ to get a periodic semi-meander $\mathbf{a} \setminus \delta \in \mathfrak{B}_{S \cup \{\tau, \tau^-\}}^{r-1}$ for $S \cup \{\tau, \tau^-\}$.

3.7. Goren–Oort cycles. We fix a pair (S, T) as before. For a periodic semi-meander \mathbf{a} for S , we define a pair $(S_{\mathbf{a}}, T_{\mathbf{a}})$ as follows: $S_{\mathbf{a}}$ is obtained by adjoining to S all end-nodes of the arcs of \mathbf{a} and $T_{\mathbf{a}}$ is obtained by adjoining to T all the *right* end-nodes (in the sense of 3.1) of the arcs of \mathbf{a} .

We now construct the *Goren–Oort cycle* $\text{Sh}_{K_p}(G_{\mathbf{S},\mathbf{T}})_{\mathbf{a}}$ associated to a periodic semi-meander \mathbf{a} for (\mathbf{S}, \mathbf{T}) . The cycle will admit an iterated \mathbb{P}^1 -bundle morphism

$$\pi_{\mathbf{a}} : \text{Sh}_{K_p}(G_{\mathbf{S},\mathbf{T}})_{\mathbf{a}} \rightarrow \text{Sh}_{K_p}(G_{\mathbf{S}_{\mathbf{a}},\mathbf{T}_{\mathbf{a}}})$$

for some appropriate subsets $\mathbf{S}_{\mathbf{a}}$ and $\mathbf{T}_{\mathbf{a}}$ of Σ_{∞} . The resulting correspondence

$$(3.7.1) \quad \text{Sh}_{K_p}(G_{\mathbf{S}_{\mathbf{a}},\mathbf{T}_{\mathbf{a}}}) \xleftarrow{\pi_{\mathbf{a}}} \text{Sh}_{K_p}(G_{\mathbf{S},\mathbf{T}})_{\mathbf{a}} \hookrightarrow \text{Sh}_{K_p}(G_{\mathbf{S},\mathbf{T}})$$

will be constructed using the unitary Shimura varieties via Construction 2.15 depending on a shift $\mathbf{t}_{\mathbf{a}} = \mathbf{t}_{\emptyset,\mathbf{a}} \in E^{\times,\text{cl}} \backslash \mathbb{A}_E^{\infty,\times} / \mathcal{O}_{E_p}^{\times}$, which is canonical up to $F^{\times,\text{cl}} \backslash \mathbb{A}_F^{\infty,\times} / \mathcal{O}_{F_p}^{\times}$, as explained in Construction 2.15.

We first define a partial order on the set of all periodic semi-meander for our fixed pair (\mathbf{S}, \mathbf{T}) by setting $\mathbf{a} \prec \mathbf{b}$ if all arcs of \mathbf{a} appears (up to graphic deformation) as arcs of \mathbf{b} . Clearly, the periodic semi-meander consisting of only semi-lines is the unique minimal element for this partial order. We will define the Goren–Oort cycles correspondence (3.7.1) inductively for the partial order above.

Suppose that we have defined this for all periodic semi-meanders $\mathbf{a}' \prec \mathbf{a}$. Now, for \mathbf{a} , among its arcs, there exists at least one arc that does not lie below any other arc in the sense of Notation 3.6. We fix such an arc δ . Let $\mathbf{a} \setminus \delta$ denote the periodic semi-meander obtained by removing the arc δ from \mathbf{a} and adjoin semi-lines to the end nodes of δ . By inductive hypothesis, we have a correspondence

$$(3.7.2) \quad \text{Sh}_{K_p}(G_{\mathbf{S}_{\mathbf{a} \setminus \delta}, \mathbf{T}_{\mathbf{a} \setminus \delta}}) \xleftarrow{\pi_{\mathbf{a} \setminus \delta}} \text{Sh}_{K_p}(G_{\mathbf{S},\mathbf{T}})_{\mathbf{a} \setminus \delta} \hookrightarrow \text{Sh}_{K_p}(G_{\mathbf{S},\mathbf{T}})$$

with shift $\mathbf{t}_{\mathbf{a} \setminus \delta}$. Let τ (resp. τ_-) denote the right (resp. left) end-node of δ . Applying Proposition 2.31(1) to the Shimura variety $\text{Sh}_{K_p}(G_{\mathbf{S}_{\mathbf{a} \setminus \delta}, \mathbf{T}_{\mathbf{a} \setminus \delta}})$, we deduce a natural \mathbb{P}^1 -bundle morphism

$$\pi_{\tau} : \text{Sh}_{K_p}(G_{\mathbf{S}_{\mathbf{a} \setminus \delta}, \mathbf{T}_{\mathbf{a} \setminus \delta}})_{\tau} \longrightarrow \text{Sh}_{K_p}(G_{\mathbf{S}_{\mathbf{a} \setminus \delta} \cup \{\tau, \tau_-\}, \mathbf{T}_{\mathbf{a} \setminus \delta} \cup \{\tau\}})$$

with some shift $\mathbf{t}_{\mathbf{a} \setminus \delta, \mathbf{a}}$ (and we fix such a choice). We define the *Goren–Oort cycle* $\text{Sh}_{K_p}(G_{\mathbf{S},\mathbf{T}})_{\mathbf{a}}$ to be

$$\text{Sh}_{K_p}(G_{\mathbf{S},\mathbf{T}})_{\mathbf{a}} := \pi_{\mathbf{a} \setminus \delta}^{-1}(\text{Sh}_{K_p}(G_{\mathbf{S}_{\mathbf{a} \setminus \delta}, \mathbf{T}_{\mathbf{a} \setminus \delta}})_{\tau}),$$

namely it fits the following commutative diagram where the square is Cartesian.

$$\begin{array}{ccccc} \text{Sh}_{K_p}(G_{\mathbf{S},\mathbf{T}})_{\mathbf{a}} & \xhookrightarrow{\quad} & \text{Sh}_{K_p}(G_{\mathbf{S},\mathbf{T}})_{\mathbf{a} \setminus \delta} & \xhookrightarrow{\quad} & \text{Sh}_{K_p}(G_{\mathbf{S},\mathbf{T}}) \\ \downarrow & & \downarrow \pi_{\mathbf{a} \setminus \delta} & & \\ \text{Sh}_{K_p}(G_{\mathbf{S}_{\mathbf{a} \setminus \delta}, \mathbf{T}_{\mathbf{a} \setminus \delta}})_{\tau} & \xhookrightarrow{\quad} & \text{Sh}_{K_p}(G_{\mathbf{S}_{\mathbf{a} \setminus \delta}, \mathbf{T}_{\mathbf{a} \setminus \delta}}) & & \\ \downarrow \pi_{\tau} & & & & \\ \text{Sh}_{K_p}(G_{\mathbf{S}_{\mathbf{a}}, \mathbf{T}_{\mathbf{a}}}) & & & & \end{array}$$

The induced correspondence

$$\text{Sh}_{K_p}(G_{\mathbf{S}_{\mathbf{a}}, \mathbf{T}_{\mathbf{a}}}) \xleftarrow{\pi_{\mathbf{a}} = \pi_{\tau} \circ \pi_{\mathbf{a} \setminus \delta}} \text{Sh}_{K_p}(G_{\mathbf{S},\mathbf{T}})_{\mathbf{a}} \hookrightarrow \text{Sh}_{K_p}(G_{\mathbf{S},\mathbf{T}})$$

has shift

$$\mathbf{t}_{\mathbf{a}} = \mathbf{t}_{\emptyset, \mathbf{a}} := \mathbf{t}_{\emptyset, \mathbf{a} \setminus \delta} \cdot \mathbf{t}_{\mathbf{a} \setminus \delta, \mathbf{a}}.$$

This completes the inductive construction of the Goren–Oort cycles.

Note that, in the process of the inductive construction, by Theorem 2.32(1), if δ' is another arc of \mathbf{a} with left (resp. right) end-nodes τ' (resp. τ'_-) that does not lie below any other arc, then we have a correspondence

$$\text{Sh}_{K_p}(G_{\mathbf{S}_{\mathbf{a}}, \mathbf{T}_{\mathbf{a}}}) \xleftarrow{\pi_{\delta'}} \text{Sh}_{K_p}(G_{\mathbf{S}_{\mathbf{a} \setminus \delta'}, \mathbf{T}_{\mathbf{a} \setminus \delta'}})_{\tau'} \hookrightarrow \text{Sh}_{K_p}(G_{\mathbf{S}_{\mathbf{a} \setminus \delta'}, \mathbf{T}_{\mathbf{a} \setminus \delta'}})$$

such that $\pi_{\delta'}$ is a \mathbb{P}^1 -bundle and, by Remark 2.16, the shift of this correspondence is

$$t_{a \setminus \delta', a} := t_a t_{a \setminus \delta'}^{-1}.$$

So inductively applying this discussion, we have, for every periodic semi-meander $\mathbf{a} \prec \mathbf{b}$, a correspondence

$$(3.7.3) \quad \mathrm{Sh}_{K_p}(G_{\mathbf{S}_a, \mathbf{T}_a}) \xleftarrow{\pi_{\mathbf{b}, \mathbf{a}}} \mathrm{Sh}_{K_p}(G_{\mathbf{S}_b, \mathbf{T}_b})_{a \setminus b} \hookrightarrow \mathrm{Sh}_{K_p}(G_{\mathbf{S}_b, \mathbf{T}_b})$$

of shift $t_{b, a} := t_a t_b^{-1}$.

We point out that a key feature of our construction is that *the dimension of fibers of $\mathrm{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})_a$ over $\mathrm{Sh}_{K_p}(G_{\mathbf{S}_a, \mathbf{T}_a})$ is the same as the codimension of $\mathrm{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})_a$ in $\mathrm{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})$, which is r .*

We fix a regular multiweight (\underline{k}, w) . Recall that $\mathcal{L}_{\mathbf{S}, \mathbf{T}}^{(\underline{k}, w)}$ denotes the automorphic ℓ -adic local system on $\mathrm{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})$. The same construction above also gives rise to an isomorphism

$$\pi_a^\sharp : \pi_a^*(\mathcal{L}_{\mathbf{S}_a, \mathbf{T}_a}^{(\underline{k}, w)}) \xrightarrow{\cong} \mathcal{L}_{\mathbf{S}, \mathbf{T}}^{(\underline{k}, w)}|_{\mathrm{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})_a}.$$

Remark 3.8. It was pointed out to us by X. Zhu that the union of all Goren–Oort cycles associated to periodic semi-meanders with r arcs is exactly the closure of certain *Newton strata* of the unitary Shimura variety, transported to the quaternionic side. In the case of Hilbert modular varieties, the relevant Newton polygon is as explained in Remark 3.8. So maybe the name “Goren–Oort” is slightly misleading, as it usually refers to stratification given by the p -torsion subgroup of the universal abelian varieties.

Example 3.9. Let F be of degree 6 over \mathbb{Q} and $\mathbf{S} = \mathbf{T} = \emptyset$. Then $\mathrm{Sh}_K(G_{\emptyset, \emptyset})$ is (the special fiber of) the Hilbert modular variety for F . Let τ_0, \dots, τ_5 denote the embeddings of \mathcal{O}_F into $\mathbb{Z}_p^{\mathrm{ur}}$ so that $\tau_i = \tau_{i \pmod{6}}$ and $\tau_{i+1} = \sigma \tau_i$. We have a universal abelian variety A over $\mathrm{Sh}_K(G_{\emptyset, \emptyset})$ equipped with an \mathcal{O}_F -action.

We consider the periodic semi-meander $\mathbf{a} = \bullet \bullet \bullet \bullet \bullet \bullet$. For each $\overline{\mathbb{F}}_p$ -point $x \in \mathrm{Sh}_K(G_{\emptyset, \emptyset})$, the Dieudonné module \mathcal{D}_x of the universal abelian variety A_x at x decomposes as $\mathcal{D}_x = \bigoplus_{i=0}^5 \mathcal{D}_{x, i}$, where \mathcal{O}_F acts on the i -th factor via τ_i . Let $V_i : \mathcal{D}_{x, i+1} \rightarrow \mathcal{D}_{x, i}$ denote the Verschiebung map for $i \in \mathbb{Z}/5\mathbb{Z}$. Then $x \in \mathrm{Sh}_K(G_{\emptyset, \emptyset})_a$ if and only if

$$V_1 \circ V_2(\mathcal{D}_{x, 3}) \subseteq p\mathcal{D}_{x, 1}, \quad V_4 \circ V_5(\mathcal{D}_{x, 0}) \subseteq p\mathcal{D}_{x, 4}, \quad \text{and } V_0 \circ V_1 \circ V_2 \circ V_3(\mathcal{D}_{x, 4}) \subseteq p^2\mathcal{D}_{x, 0}.$$

In fact, these inclusions are forced to be equalities. In this case, $\mathrm{Sh}_{K_p}(G_{\emptyset, \emptyset})_a$ is a collections of “iterated \mathbb{P}^1 -bundles” parametrized by the discrete Shimura variety $\mathrm{Sh}_K(G_{\Sigma_\infty, \{\tau_2, \tau_3, \tau_5\}})$. Moreover, one can prove that each geometric connected component is isomorphic to the product of \mathbb{P}^1 with the projective bundle $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-p) \oplus \mathcal{O}_{\mathbb{P}^1}(p))$ over \mathbb{P}^1 .¹⁸

4. COHOMOLOGY OF GOREN–OORT CYCLES

Let $\mathrm{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})$ be the special fiber of a quaternionic Shimura variety as in Section 2.2. Using Gysin maps, the cohomology of the Goren–Oort cycles gives rise to part of the cohomology of the big Shimura variety $\mathrm{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})$.

4.1. Generalities on étale cohomology. We recall first some generalities on Gysin maps and étale cohomology of iterated \mathbb{P}^1 -bundles. Let ℓ be a fixed prime number, and k be an algebraically closed field of characteristic different from ℓ .

Consider a closed immersion $i : Y \hookrightarrow X$ of smooth varieties over k of codimension r . The functor of direct image i_* has a right adjoint, denoted by $i^!$. For an ℓ -adic étale sheaf \mathcal{F} on X , $i^!\mathcal{F}$ is the sheaf of sections of \mathcal{F} with support in Y . This is a left exact functor, and let $R^q i^!$ denote its q -th derived

¹⁸More canonically, it is the projective bundle of a rank two bundle E over \mathbb{P}^1 which sits inside an exact sequence $0 \rightarrow \mathcal{O}_{\mathbb{P}^1}(p) \rightarrow E \rightarrow \mathcal{O}_{\mathbb{P}^1}(-p) \rightarrow 0$. The extension splits (not canonically though).

functor. Then by the relative cohomological purity [SGA4, XVI, Théorème 3.7], we have $R^q i^! \overline{\mathbb{Q}}_\ell = 0$ for $q \neq 2r$, and a canonical isomorphism $R^{2r} i^! \overline{\mathbb{Q}}_\ell \xrightarrow{\cong} \overline{\mathbb{Q}}_\ell(-r)$. Explicitly, the inverse isomorphism $\overline{\mathbb{Q}}_\ell \xrightarrow{\cong} R^{2r} i^! \overline{\mathbb{Q}}_\ell(r)$ is given by the fundamental class $\text{cl}_Y \in H_{\text{et}, Y}^{2r}(X, \overline{\mathbb{Q}}_\ell(r)) \cong H_{\text{et}}^0(Y, R^{2r} i^! \overline{\mathbb{Q}}_\ell)$ of Y . Now for any ℓ -adic lisse sheaf \mathcal{F} on X , we define the Gysin map as the composite

$$(4.1.1) \quad \text{Gysin}: H_{\text{et}}^q(Y, i^* \mathcal{F}) \xrightarrow{\cup \text{cl}_Y} H_{\text{et}, Y}^{q+2r}(X, \mathcal{F}) \rightarrow H_{\text{et}}^{q+2r}(X, \mathcal{F}),$$

where the second map is the canonical morphism from cohomology supported in Y to the usual cohomology group.

Let $\pi: X \rightarrow Y$ be an r -th iterated \mathbb{P}^1 -bundle of proper and smooth k -varieties, i.e. π admits a factorization

$$(4.1.2) \quad \pi: X_0 := X \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_2} X_2 \rightarrow \cdots \xrightarrow{\pi_r} X_r := Y,$$

where each $\pi_i: X_{i-1} \rightarrow X_i$ is a \mathbb{P}^1 -fibration for $1 \leq i \leq r$. Then the trace map

$$\text{Tr}_\pi: R^{2r} \pi_* (\overline{\mathbb{Q}}_\ell(r)) \xrightarrow{\cong} \overline{\mathbb{Q}}_\ell$$

is an isomorphism. We denote by $\text{cl}_\pi \in H^0(Y, R^{2r} \pi_* \overline{\mathbb{Q}}_\ell(r))$ with $\text{Tr}_\pi(\text{cl}_\pi) = 1$, and call it *fundamental class of the fibration* π . For any $\overline{\mathbb{Q}}_\ell$ -lisse sheaf \mathcal{F} on Y and any integer $q \geq 0$, it induces a map

$$\pi_!: H_{\text{et}}^q(X, \pi^* \mathcal{F}(r)) \rightarrow H_{\text{et}}^{q-2r}(Y, \mathcal{F} \otimes R^{2r} \pi_* (\overline{\mathbb{Q}}_\ell(r))) \xrightarrow{\text{Tr}_\pi} H_{\text{et}}^{q-2r}(Y, \mathcal{F}),$$

where the first morphism comes from the Leray spectral sequence $E_2^{a,b} = H_{\text{et}}^a(Y, R^b \pi_* \pi^* \mathcal{F}(r)) \Rightarrow H_{\text{et}}^{a+b}(X, \pi^* \mathcal{F}(r))$. Explicitly, $\pi_!$ admits the following description. Put $\pi_{[0,i]} := \pi_i \circ \pi_{i-1} \circ \cdots \circ \pi_1$ for $1 \leq i \leq r$. Let $\mathcal{O}_{\pi_i}(1)$ be the tautological quotient line bundle of the \mathbb{P}^1 -bundle π_i , and $c_1(\mathcal{O}_{\pi_i}(1)) \in H_{\text{et}}^2(X_{i-1}, \overline{\mathbb{Q}}_\ell(1))$ be its first Chern class. Put $\xi_i = \pi_{[0,i-1]}^* c_1(\mathcal{O}_{\pi_i}(1)) \in H_{\text{et}}^2(X, \overline{\mathbb{Q}}_\ell(1))$. By induction on r , one deduces easily from [SGA5, Corollaire 2.2.6] a decomposition

$$H_{\text{et}}^q(X, \pi^* \mathcal{F}(r)) \cong \bigoplus_{0 \leq j \leq r} \left(\bigoplus_{1 \leq i_1 < \cdots < i_j \leq r} \pi^* H_{\text{et}}^{q-2j}(Y, \mathcal{F}(r-j)) \cup \xi_{i_1} \cup \cdots \cup \xi_{i_j} \right).$$

Then for an element $x = \sum_j \sum_{1 \leq i_1 < \cdots < i_j \leq r} \pi^*(y_{i_1, \dots, i_j}) \cup \xi_{i_1} \cup \cdots \cup \xi_{i_j}$, one has

$$(4.1.3) \quad \pi_!(x) = y_{1, \dots, r}.$$

In particular, the fundamental class cl_π is the image of $\xi_1 \cup \cdots \cup \xi_r$ in $H_{\text{et}}^0(X, R^{2r} \pi_* \overline{\mathbb{Q}}_\ell(r))$.

4.2. Gysin and restriction maps. We keep the notation of Section 3.7. The pair of morphisms (π_a, π_a^\sharp) induces the following sequence of natural homomorphisms, whose composition we denote by Gys_a ,

$$\begin{aligned} H_{\text{et}}^{d-2r}(\text{Sh}_{K_p}(G_{\mathbb{S}_a, \mathbb{T}_a})_{\overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbb{S}_a, \mathbb{T}_a}^{(k,w)}) &\xrightarrow{\pi_a^*, \cong} H_{\text{et}}^{d-2r}(\text{Sh}_{K_p}(G_{\mathbb{S}, \mathbb{T}})_{a, \overline{\mathbb{F}}_p}, \pi_a^*(\mathcal{L}_{\mathbb{S}_a, \mathbb{T}_a}^{(k,w)})) \\ &\xrightarrow{\pi_a^\sharp, \cong} H_{\text{et}}^{d-2r}(\text{Sh}_{K_p}(G_{\mathbb{S}, \mathbb{T}})_{a, \overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbb{S}, \mathbb{T}}^{(k,w)}|_{\text{Sh}_{K_p}(G_{\mathbb{S}, \mathbb{T}})_a}) \\ &\xrightarrow{\text{Gysin}, (4.1.1)} H_{\text{et}}^d(\text{Sh}_{K_p}(G_{\mathbb{S}, \mathbb{T}})_{\overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbb{S}, \mathbb{T}}^{(k,w)}(r)). \end{aligned}$$

We can also consider the dual picture, defining the morphism Res_a to be the composition of the following homomorphisms:

$$\begin{aligned} \text{Res}_a: H_{\text{et}}^d(\text{Sh}_{K_p}(G_{\mathbb{S}, \mathbb{T}})_{\overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbb{S}, \mathbb{T}}^{(k,w)}(r)) &\xrightarrow{\text{Restriction}} H_{\text{et}}^d(\text{Sh}_{K_p}(G_{\mathbb{S}, \mathbb{T}})_{a, \overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbb{S}, \mathbb{T}}^{(k,w)}|_{\text{Sh}_{K_p}(G_{\mathbb{S}, \mathbb{T}})_a}(r)) \\ &\xrightarrow{(\pi_a^\sharp)^{-1}, \cong} H_{\text{et}}^d(\text{Sh}_{K_p}(G_{\mathbb{S}, \mathbb{T}})_{a, \overline{\mathbb{F}}_p}, \pi_a^* \mathcal{L}_{\mathbb{S}_a, \mathbb{T}_a}^{(k,w)}(r)) \\ &\xrightarrow{\pi_{a,!}} H_{\text{et}}^{d-2r}(\text{Sh}_{K_p}(G_{\mathbb{S}_a, \mathbb{T}_a})_{\overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbb{S}_a, \mathbb{T}_a}^{(k,w)}). \end{aligned}$$

It is clear from the construction that both morphisms $\text{Gys}_{\mathbf{a}}$ and $\text{Res}_{\mathbf{a}}$ are equivariant for the action of the tame Hecke algebra $\overline{\mathbb{Q}}_{\ell}[K^p \backslash G(\mathbb{A}^{\infty, p})/K^p]$.

The following theorem is the key to proving our main result. We defer its proof to the next section.

Theorem 4.3. *Fix $\pi \in \mathcal{A}_{(k, w)}$, and fix a choice of system of shifts $\mathbf{t}_{\mathbf{a}}$ of the correspondences $\text{Sh}_{K_p}(G_{\mathbf{S}_{\mathbf{a}}, \mathbf{T}_{\mathbf{a}}}) \xleftarrow{\pi_{\mathbf{a}}} \text{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})_{\mathbf{a}} \hookrightarrow \text{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})$ as in Subsection 3.7. For $\mathbf{a}, \mathbf{b} \in \mathfrak{B}_{\mathbf{S}}^r$, we have the following description of the composition*

$$\begin{aligned} H_{\text{et}}^{d-2r}(\text{Sh}_{K_p}(G_{\mathbf{S}_{\mathbf{b}}, \mathbf{T}_{\mathbf{b}}})_{\overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{S}_{\mathbf{b}}, \mathbf{T}_{\mathbf{b}}}^{(k, w)}) &\xrightarrow{\text{Gys}_{\mathbf{b}}} H_{\text{et}}^d(\text{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})_{\overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{S}, \mathbf{T}}^{(k, w)}(r)) \\ &\xrightarrow{\text{Res}_{\mathbf{a}}} H_{\text{et}}^{d-2r}(\text{Sh}_{K_p}(G_{\mathbf{S}_{\mathbf{a}}, \mathbf{T}_{\mathbf{a}}})_{\overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{S}_{\mathbf{a}}, \mathbf{T}_{\mathbf{a}}}^{(k, w)}). \end{aligned}$$

- (1) When $\langle \mathbf{a} | \mathbf{b} \rangle = 0$, the π -isotypical component of the composed map $\text{Res}_{\mathbf{a}} \circ \text{Gys}_{\mathbf{b}}$ factors through the π -isotypical component of the cohomology group $H_{\text{et}}^{d-2(r+1)}(\text{Sh}_{K_p}(G_{\mathbf{S}', \mathbf{T}'})_{\overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{S}', \mathbf{T}'}^{(k, w)})(-1)$ of some quaternionic Shimura variety of dimension $d - 2(r + 1)$ with $\#\mathbf{T}' = \#\mathbf{T} + (r + 1)$ and \mathbf{S}' having the same set of finite places as \mathbf{S} .
- (2) When $r < \frac{d}{2}$ and $\langle \mathbf{a} | \mathbf{b} \rangle = (-2)^{m_0(\mathbf{a}, \mathbf{b})} v^{m_v(\mathbf{a}, \mathbf{b})}$, we can define the induced link $\eta_{\mathbf{S}_{\mathbf{a}}, \mathbf{S}_{\mathbf{b}}} : \mathbf{S}_{\mathbf{a}} \rightarrow \mathbf{S}_{\mathbf{b}}$ as in Subsection 3.2. Then there exists a normalized link morphism in the sense of Subsection 2.28

$$\eta_{\mathbf{S}_{\mathbf{a}}, \mathbf{S}_{\mathbf{b}}, (z)}^* : H_{\text{et}}^{d-2r}(\text{Sh}_{K_p}(G_{\mathbf{S}_{\mathbf{b}}, \mathbf{T}_{\mathbf{b}}})_{\overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{S}_{\mathbf{b}}, \mathbf{T}_{\mathbf{b}}}^{(k, w)}) \rightarrow H_{\text{et}}^{d-2r}(\text{Sh}_{K_p}(G_{\mathbf{S}_{\mathbf{a}}, \mathbf{T}_{\mathbf{a}}})_{\overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{S}_{\mathbf{a}}, \mathbf{T}_{\mathbf{a}}}^{(k, w)}),$$

associated to $\eta_{\mathbf{S}_{\mathbf{a}}, \mathbf{S}_{\mathbf{b}}}$ with shift $\mathbf{t}_{\mathbf{a}} \mathbf{t}_{\mathbf{b}}^{-1}$ and indentation degree

$$z = \begin{cases} \ell(\mathbf{a}) - \ell(\mathbf{b}) & \text{if } \mathfrak{p} \text{ splits in } E/F, \\ 0 & \text{if } \mathfrak{p} \text{ is inert in } E/F. \end{cases}$$

Moreover, we have an equality

$$\text{Res}_{\mathbf{a}} \circ \text{Gys}_{\mathbf{b}} = (-2)^{m_0(\mathbf{a}, \mathbf{b})} \cdot p^{(\ell(\mathbf{a}) + \ell(\mathbf{b}))/2} \eta_{\mathbf{S}_{\mathbf{a}}, \mathbf{S}_{\mathbf{b}}, (z)}^*.$$

- (3) When $r = \frac{d}{2}$ and $\langle \mathbf{a} | \mathbf{b} \rangle = (-2)^{m_0(\mathbf{a}, \mathbf{b})} T^{m_T(\mathbf{a}, \mathbf{b})}$, we have

$$\text{Res}_{\mathbf{a}} \circ \text{Gys}_{\mathbf{b}} = (-2)^{m_0(\mathbf{a}, \mathbf{b})} \cdot p^{(\ell(\mathbf{a}) + \ell(\mathbf{b}))/2} (T_{\mathfrak{p}}/p^{g/2})^{m_T(\mathbf{a}, \mathbf{b})} \circ \eta_{\mathbf{S}_{\mathbf{a}}, \mathbf{S}_{\mathbf{b}}, (z)}^*,$$

where $\eta_{\mathbf{S}_{\mathbf{a}}, \mathbf{S}_{\mathbf{b}}}$ is the trivial link from $\mathbf{S}_{\mathbf{a}}$ to $\mathbf{S}_{\mathbf{b}}$ and

$$\eta_{\mathbf{S}_{\mathbf{a}}, \mathbf{S}_{\mathbf{b}}, (z)}^* : H_{\text{et}}^{d-2r}(\text{Sh}_{K_p}(G_{\mathbf{S}_{\mathbf{b}}, \mathbf{T}_{\mathbf{b}}})_{\overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{S}_{\mathbf{b}}, \mathbf{T}_{\mathbf{b}}}^{(k, w)}) \longrightarrow H_{\text{et}}^{d-2r}(\text{Sh}_{K_p}(G_{\mathbf{S}_{\mathbf{a}}, \mathbf{T}_{\mathbf{a}}})_{\overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{S}_{\mathbf{a}}, \mathbf{T}_{\mathbf{a}}}^{(k, w)})$$

is the associated normalized link morphism with shift $\mathbf{t}_{\mathbf{a}} \mathbf{t}_{\mathbf{b}}^{-1} \varpi_{\mathfrak{q}}^{-m_T(\mathbf{a}, \mathbf{b})}$ and indentation degree $z = \ell(\mathbf{a}) - \ell(\mathbf{b}) - m_T(\mathbf{a}, \mathbf{b})g$.

We now assume Theorem 4.3 and deduce the main theorem of this paper.

Theorem 4.4. *Fix an positive integer $r \leq \frac{d}{2}$, and keep the notation of Theorem 4.3.*

- (1) *For each periodic semi-meander $\mathbf{a} \in \mathfrak{B}_{\mathbf{S}}^r$, the Goren–Oort cycle $\text{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})_{\mathbf{a}}$ of the Shimura variety $\text{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})$ is a subvariety of codimension r , stable under the action of the tame Hecke action of $G(\mathbb{A}^{\infty, p})$. Moreover, it admits a natural proper smooth morphism*

$$\pi_{\mathbf{a}} : \text{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})_{\mathbf{a}} \rightarrow \text{Sh}_{K_p}(G_{\mathbf{S}_{\mathbf{a}}, \mathbf{T}_{\mathbf{a}}})$$

to another quaternionic Shimura variety (in characteristic p), such that the fibers of $\pi_{\mathbf{a}}$ are r -times iterated \mathbb{P}^1 -bundles. The morphism is equivariant for the tame Hecke action.

- (2) We fix a cuspidal automorphic representation $\pi \in \mathcal{A}_{(\underline{k}, w)}$ so that its associated Galois representation ρ_π is unramified at p . Let α_π and β_π denote the (generalized) eigenvalues of $\rho_{\pi, \mathfrak{p}}(\text{Frob}_{p^g})$. Suppose that α_π/β_π is not a $2n$ -th root of unity for any $n \leq d$ so that $\alpha_\pi^{2i}\beta_\pi^{2(d-i)}$ are distinct from each other for $1 \leq i \leq d$. Then the action of $\text{Frob}_{p^{2g}}$ on the generalized eigenspace of $H_{\text{et}}^d(\text{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})_{\overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{S}, \mathbf{T}}^{(k, w)}(r))[\pi]$ with eigenvalue $\alpha_\pi^{2(d-r)}\beta_\pi^{2r}(\alpha_\pi\beta_\pi/p^g)^{2\#\mathbf{T}}p^{-2gr}$ is semisimple (so that the generalized eigenspace is a genuine eigenspace), and the direct sum of the Gysin morphisms

$$(4.4.1) \quad \bigoplus_{\mathfrak{a} \in \mathfrak{B}_{\mathbf{S}}^r} H_{\text{et}}^{d-2r}(\text{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})_{\mathfrak{a}, \overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{S}, \mathbf{T}}^{(k, w)}(r))[\pi] \xrightarrow{\sum_{\mathfrak{a}} \text{Gys}_{\mathfrak{a}}} H_{\text{et}}^d(\text{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})_{\overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{S}, \mathbf{T}}^{(k, w)}(r))[\pi]$$

induces an isomorphism on the $\text{Frob}_{p^{2g}}$ -eigenspaces with eigenvalue $\alpha_\pi^{2(d-r)}\beta_\pi^{2r}(\alpha_\pi\beta_\pi/p^g)^{2\#\mathbf{T}}p^{-2gr}$.

- (2') Keep the notation in (2) but assume that $r = \frac{d}{2}$ (so d is even) and $(\underline{k}, w) = \underline{2}$. Suppose that α_π/β_π is not an $2n$ -th root of unity for $n \leq \frac{d}{2}$. Then the $\text{Frob}_{p^{2g}}$ -invariant subspace of $H_{\text{et}}^d(\text{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(\frac{d}{2}))[\pi]$ is generated by the cycle classes of $\text{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})_{\mathfrak{a}}$ for $\mathfrak{a} \in \mathfrak{B}_{\mathbf{S}}^{d/2}$.

Proof. Statement (1) follows from the construction of Goren–Oort cycles in Subsection 3.7. (2') is clearly a special case of (2). We now focus on the proof of (2). By Proposition 2.26, the morphism (4.4.1) is the same as

$$(4.4.2) \quad \bigoplus_{\mathfrak{a} \in \mathfrak{B}_{\mathbf{S}}^r} \rho_{\pi, \mathfrak{p}}^{\otimes(d-2r)} \otimes (\det \rho_{\pi, \mathfrak{p}}(1))^{\otimes(\#\mathbf{T}+r)} \longrightarrow \rho_{\pi, \mathfrak{p}}^{\otimes d} \otimes (\det \rho_{\pi, \mathfrak{p}}(1))^{\otimes \#\mathbf{T}}(r).$$

Thus the generalized eigenspace for the action of $\text{Frob}_{p^{2g}}$ with eigenvalue

$$(4.4.3) \quad \alpha_\pi^{2(d-2r)}(\alpha_\pi\beta_\pi/p^g)^{2(\#\mathbf{T}+r)} = \alpha_\pi^{2(d-r)}\beta_\pi^{2r}(\alpha_\pi\beta_\pi/p^g)^{2\#\mathbf{T}}p^{-2gr}$$

has dimension exactly equal to $\binom{d}{r}$ for both sides of (4.4.2); and the generalized eigenspace on left hand side is a genuine eigenspace (since it is the direct sum of $\binom{d}{r}$ -copies of one-dimensional generalized eigenspace). Here, we used the assumption on the ratio of Satake parameters. Thus, the proof of (2) and (2') will be finished if we show that (4.4.2) is injective on the corresponding generalized eigenspace.

We consider the composition of the Gysin morphisms (4.4.1) with the Restriction morphisms:

$$(4.4.4) \quad \bigoplus_{\mathfrak{b} \in \mathfrak{B}_{\mathbf{S}}^r} H_{\text{et}}^{d-2r}(\text{Sh}_{K_p}(G_{\mathbf{S}_{\mathfrak{b}}, \mathbf{T}_{\mathfrak{b}}})_{\overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{S}_{\mathfrak{b}}, \mathbf{T}_{\mathfrak{b}}}^{(k, w)}(r))[\pi] \xrightarrow{\sum \text{Gys}_{\mathfrak{b}}} H_{\text{et}}^d(\text{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})_{\overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{S}, \mathbf{T}}^{(k, w)}(r))[\pi] \\ \xrightarrow{\oplus \text{Res}_{\mathfrak{a}}} \bigoplus_{\mathfrak{a} \in \mathfrak{B}_{\mathbf{S}}^r} H_{\text{et}}^{d-2r}(\text{Sh}_{K_p}(G_{\mathbf{S}_{\mathfrak{a}}, \mathbf{T}_{\mathfrak{a}}})_{\overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{S}_{\mathfrak{a}}, \mathbf{T}_{\mathfrak{a}}}^{(k, w)}(r))[\pi].$$

Here, we switched the first sum from over \mathfrak{a} (as in (4.4.1)) to over \mathfrak{b} . Taking the generalized eigenspace for $\text{Frob}_{p^{2g}}$ acting on (4.4.4) with the eigenvalue (4.4.3) and using the description Proposition 2.26, we arrive at the following linear map

$$(4.4.5) \quad \bigoplus_{\mathfrak{b} \in \mathfrak{B}_{\mathbf{S}}^r} \overline{\mathbb{Q}}_\ell \rightarrow \bigoplus_{\mathfrak{a} \in \mathfrak{B}_{\mathbf{S}}^r} \overline{\mathbb{Q}}_\ell$$

of vector spaces. Choosing a basis for each one-dimensional summand, this map is represented by a $\binom{d}{r} \times \binom{d}{r}$ -matrix A with coefficients in $\overline{\mathbb{Q}}_\ell$.

We explain how this matrix A is related to the Gram matrix for the periodic semi-meanders. We first normalize A by multiplying A on both sides by the same diagonal matrix, whose diagonal component is $p^{-\ell(\mathfrak{a})/2}$ at \mathfrak{a} . This will kill the auxiliary factor $p^{(\ell(\mathfrak{a})+\ell(\mathfrak{b}))/2}$ in Theorem 4.3. Let B

denote the product matrix. We will prove that

$$\det B = \begin{cases} \det \mathfrak{G}_S^{d/2}|_{T^2=T_p^n}, & \text{if } r = d/2 \\ \det \mathfrak{G}_S^r|_{v^g=\eta_{\text{univ}}^*}, & \text{if } r < d/2, \end{cases}$$

where $|_{T^2=T_p^n}$ and $|_{v^g=\eta_{\text{univ}}^*}$ are formal substitutions, and T_p^n and η_{univ}^* are some formal symbols we define later.

We first compare the entries of B with the entries of \mathfrak{G}_S^r when $\langle \mathfrak{a}|\mathfrak{b} \rangle = 0$. In this case, by Theorem 4.3(1), the π -isotypical component of $\text{Res}_{\mathfrak{a}} \circ \text{Gys}_{\mathfrak{b}}$ factors through

$$H^{d-2(r+1)}(\text{Sh}_{K_p}(G_{S',T'})_{\overline{\mathbb{F}}_p}, \mathcal{L}_{S',T'}^{(k,w)})(-1)[\pi]$$

for some quaternionic Shimura variety $\text{Sh}_{K_p}(G_{S',T'})$ of dimension $d - 2(r + 1)$. Thanks to the assumption on the ratio of Satake parameters, we see that $\alpha_\pi^{2i} \beta_\pi^{2(d-i)}$ are distinct. The cohomology group above does not contain any generalized $\text{Frob}_{p^{2g}}$ -eigenspaces with eigenvalue (4.4.3). Thus the $(\mathfrak{a}, \mathfrak{b})$ -entry of B is zero.

We separate the discussion for $r < \frac{d}{2}$ and $r = \frac{d}{2}$. First, suppose that $r < \frac{d}{2}$. A subtle point of our argument is that we can not directly identify the matrix B with \mathfrak{G}_S^r entry by entry, because there is no canonical choice of basis on each of the factor of (4.4.5). The proof resembles to that of Theorem 3.5. The determinant of B is equal to the sum over all permutations s of the set \mathfrak{B}_S^r , of the product of the signature of s , and, for every cycle $(\mathfrak{a}_1 \cdots \mathfrak{a}_t)$ of the permutation s , the product

$$(4.4.6) \quad p^{-(\ell(\mathfrak{a}_1) + \cdots + \ell(\mathfrak{a}_t))} \cdot (\text{Res}_{\mathfrak{a}_1} \circ \text{Gys}_{\mathfrak{a}_t}) \cdot (\text{Res}_{\mathfrak{a}_t} \circ \text{Gys}_{\mathfrak{a}_{t-1}}) \cdots (\text{Res}_{\mathfrak{a}_2} \circ \text{Gys}_{\mathfrak{a}_1}).$$

Let $m_0 = m_0(\mathfrak{a}_1, \mathfrak{a}_2) + \cdots + m_0(\mathfrak{a}_t, \mathfrak{a}_1)$ be the sum of total number of contractible loops in the diagrams $D(\mathfrak{a}_1, \mathfrak{a}_2)$, $D(\mathfrak{a}_2, \mathfrak{a}_3)$, \dots , $D(\mathfrak{a}_t, \mathfrak{a}_1)$. Then by Theorem 4.3(2), (4.4.6) is of the form $(-2)^{m_0}$ times the following composition of link morphisms on the cohomology groups:

$$(4.4.7) \quad \eta_{S_{\mathfrak{a}_1}, S_{\mathfrak{a}_2}, z(\mathfrak{a}_1, \mathfrak{a}_2)}^* \circ \eta_{S_{\mathfrak{a}_2}, S_{\mathfrak{a}_3}, z(\mathfrak{a}_2, \mathfrak{a}_3)}^* \circ \cdots \circ \eta_{S_{\mathfrak{a}_t}, S_{\mathfrak{a}_1}, z(\mathfrak{a}_t, \mathfrak{a}_1)}^*,$$

of shift $\prod_{i=1}^{t-1} (t_{\mathfrak{a}_i} t_{\mathfrak{a}_{i+1}}^{-1}) t_{\mathfrak{a}_t} t_{\mathfrak{a}_1}^{-1} = 1$ and indentation degree

$$\sum_{i=1}^{t-1} z(\mathfrak{a}_i, \mathfrak{a}_{i+1}) + z(\mathfrak{a}_t, \mathfrak{a}_1) = \begin{cases} \sum_{i=1}^{t-1} (\ell(\mathfrak{a}_i) - \ell(\mathfrak{a}_{i+1})) + \ell(\mathfrak{a}_n) - \ell(\mathfrak{a}_1) = 0, & \text{if } \mathfrak{p} \text{ splits in } E/F, \\ 0 + \cdots + 0 = 0, & \text{if } \mathfrak{p} \text{ is inert in } E/F. \end{cases}$$

So this composition (4.4.7) is the same link morphism associated to some n -th power of the *fundamental link* $\eta_{S_{\mathfrak{a}_1}}$ for $S_{\mathfrak{a}_1}$, with trivial shift and indentation degree 0 (no matter \mathfrak{p} splits or not in E/F).

Note that the link morphism $(\eta_{S_{\mathfrak{a}_1}}^n)_{(0)}^*$ acting on the one-dimensional $\text{Frob}_{p^{2g}}$ -eigenspace

$$(4.4.8) \quad (H_{\text{et}}^{d-2r}(\text{Sh}_{K_p}(G_{S_{\mathfrak{a}_1}, T_{\mathfrak{a}_1}}), \mathcal{L}_{S_{\mathfrak{a}_1}, T_{\mathfrak{a}_1}}^{(k,w)}))[\pi])^{\text{Frob}_{p^{2g}} = \alpha_\pi^{2(d-2r)} (\alpha_\pi \beta_\pi / p^g)^{2(\#T+r)}}$$

is just the multiplication by a scalar which we denote by $\lambda_{\mathfrak{a}_1, n}$. We claim that $\lambda_{\mathfrak{a}_1, n}$ *does not depend on* $\mathfrak{a}_1 \in \mathfrak{B}_S^r$. Indeed, for $\mathfrak{a}, \mathfrak{a}' \in \mathfrak{B}_S^r$ with $\langle \mathfrak{a}|\mathfrak{a}' \rangle \neq 0$, Theorem 4.3(2) gives a normalized link morphism

$$\eta_{S_{\mathfrak{a}}, S_{\mathfrak{a}'}, (z)}^* : H_{\text{et}}^{d-2r}(\text{Sh}_{K_p}(G_{S_{\mathfrak{a}'}, T_{\mathfrak{a}'}}), \mathcal{L}_{S_{\mathfrak{a}'}, T_{\mathfrak{a}'}}^{(k,w)}) \rightarrow (\text{Sh}_{K_p}(G_{S_{\mathfrak{a}}, T_{\mathfrak{a}}}), \mathcal{L}_{S_{\mathfrak{a}}, T_{\mathfrak{a}}}^{(k,w)})$$

with some indentation degree z and some shift, then

$$(\eta_{S_{\mathfrak{a}}}^n)_{(0)}^* = \eta_{S_{\mathfrak{a}}, S_{\mathfrak{a}'}, (z)}^* \circ (\eta_{S_{\mathfrak{a}'}}^n)_{(0)}^* \circ (\eta_{S_{\mathfrak{a}}, S_{\mathfrak{a}'}, (z)}^*)^{-1}$$

provided one of $(\eta_{S_{\mathfrak{a}}}^n)_{(0)}^*$ or $(\eta_{S_{\mathfrak{a}'}}^n)_{(0)}^*$ exists. When this happens, we must have $\lambda_{\mathfrak{a}, n} = \lambda_{\mathfrak{a}', n}$. For general \mathfrak{a} and \mathfrak{a}' , we can always find $\mathfrak{a}_1 = \mathfrak{a}, \dots, \mathfrak{a}' = \mathfrak{a}_t \in \mathfrak{B}_S^r$ such that $\langle \mathfrak{a}_i|\mathfrak{a}_{i+1} \rangle \neq 0$. So if for some n the link morphism $(\eta_{S_{\mathfrak{a}}}^n)^*$ exists, then it does not depend on \mathfrak{a} . In the sequel, we put $\lambda_n = \lambda_{\mathfrak{a}, n}$ as long as $(\eta_{S_{\mathfrak{a}}}^n)_{(0)}^*$ exists for some $\mathfrak{a} \in \mathfrak{B}_S^r$. The element λ_n is clearly multiplicative in n .

We can thus introduce the formal symbol η_{univ}^* such that $(\eta_{\text{univ}}^*)^n = \lambda_n$ whenever $(\eta_{\mathbf{S}_a}^n)_{(0)}^*$ exists for an integer n . Comparing this computation with $\det \mathfrak{G}_s^r$ in the proof of Theorem 3.5, we see that $\det B$ is obtained by replacing every v^g in $\det \mathfrak{G}_s^r$ by η_{univ}^* . By Theorem 3.5, this means that

$$\det B = \pm (\eta_{\text{univ}}^* - (\eta_{\text{univ}}^*)^{-1})^{2t_{d,r}}.$$

In particular, $(\eta_{\text{univ}}^*)^2$ appears in the determinant and hence $(\eta_{\mathbf{S}_a}^2)_{(0)}^*$ exists.

Finally, it follows from Proposition 2.27 that $(\eta_{\text{univ}}^*)^{2(d-2r)} = \lambda_{2(d-2r)} = (\alpha_\pi/\beta_\pi)^{d-2r}$. Our assumption implies that $(\alpha_\pi/\beta_\pi)^{d-2r} \neq 1$; so $(\eta_{\text{univ}}^*)^2 \neq 1$ and hence $\det B \neq 0$. This concludes (2).

We now treat the case of $r = \frac{d}{2}$. Similarly to the discussion above, $\det B$ is equal to the sum over all permutations s of the set \mathfrak{B}_s^r , of the product of the signature of s , and, for every cycle $(\mathbf{a}_1 \cdots \mathbf{a}_t)$ of the permutation s , the product (4.4.6). By Theorem 4.3(3), (4.4.6) in this case is of the form $(-2)^{m_0} \cdot (T_{\mathfrak{p}}/p^{g/2})^{m_T}$ times the link morphism from $(\mathbf{S}_{\mathbf{a}_1}, \mathbf{T}_{\mathbf{a}_1})$ to itself with shift $\prod_{i=1}^{t-1} (t_{\mathbf{a}_i} t_{\mathbf{a}_{i+1}}^{-1} \varpi_{\bar{\mathbf{q}}}^{-m_T(\mathbf{a}_i, \mathbf{a}_{i+1})}) t_{\mathbf{a}_t} t_{\mathbf{a}_1}^{-1} \varpi_{\bar{\mathbf{q}}}^{-m_T(\mathbf{a}_t, \mathbf{a}_1)} = \varpi_{\bar{\mathbf{q}}}^{-m_T}$ and indentation degree

$$\sum_{i=1}^t (\ell(\mathbf{a}_i) - \ell(\mathbf{a}_{i-1}) - m_{T,i}g) = -m_T g.$$

Here $m_0 = m_0(\mathbf{a}_1, \mathbf{a}_2) + \cdots + m_0(\mathbf{a}_t, \mathbf{a}_1)$ (resp. $m_T = m_T(\mathbf{a}_1, \mathbf{a}_2) + \cdots + m_T(\mathbf{a}_t, \mathbf{a}_1)$) is the total number of contractible (resp. non-contractible) loops in $D(\mathbf{a}_1, \mathbf{a}_2), \dots, D(\mathbf{a}_t, \mathbf{a}_1)$. By Example 2.24 and the uniqueness of link morphisms (Lemma 2.22), this link morphism is equal to the one associated to $S_{\bar{\mathbf{q}}}^{-m_T/2}$ with shift $\varpi_{\bar{\mathbf{q}}}^{-m_T}$. This in particular says that m_T is even. By the second part of Example 2.24, we see that this link morphism is exactly $S_{\bar{\mathbf{p}}}^{-m_T/2}$. Therefore, (4.4.6) is given by

$$(-2)^{m_0} (T_{\mathfrak{p}}/p^{g/2})^{m_T} (S_{\bar{\mathbf{p}}})^{-m_T/2} = (-2)^{m_0} ((\alpha_\pi + \beta_\pi)^2 / (\alpha_\pi \beta_\pi))^{m_T/2}.$$

Comparing this with the computation of $\det \mathfrak{G}_s^r$, we see that $\det B$ is nothing but replacing every T^2 by $T_{\mathfrak{p}}^n := (\alpha_\pi + \beta_\pi)^2 / \alpha_\pi \beta_\pi$. By Theorem 3.5, we see that

$$\det B = \pm ((\alpha_\pi + \beta_\pi)^2 / \alpha_\pi \beta_\pi - 4)^{t_{d,d/2}} = \pm ((\alpha_\pi - \beta_\pi) / \alpha_\pi \beta_\pi)^{t_{d,d/2}}.$$

It is nonzero as long as $\alpha_\pi \neq \beta_\pi$.¹⁹ This concludes the proof of Theorem 4.4. \square

Before giving a more detailed discussion of the case $\alpha_\pi = \beta_\pi$, we first give some general remarks.

Remark 4.5. (1) We discuss the possibility of generalizing this main theorem to the case when p is only assumed to be unramified, namely, $p\mathcal{O}_F = \mathfrak{p}_1 \cdots \mathfrak{p}_h$. In this case, one can construct the twisted partial Frobenius $\mathfrak{F}_{\mathfrak{p}_i}''$ for each prime ideal \mathfrak{p}_i as in [TX13⁺a, §3.22]. Roughly speaking, on the level of moduli space, this is to send the abelian variety A to $A / \text{Ker}_{\mathfrak{p}_i^2} \otimes_{\mathcal{O}_F} \mathfrak{p}_i$, where $\text{Ker}_{\mathfrak{p}_i^2}$ is the \mathfrak{p}_i -component of the kernel of $\text{Fr}^2 : A \rightarrow A^{(p^2)}$.

Suppose that one can describe the action of $\mathfrak{F}_{\mathfrak{p}_i}''$ on the cohomology of the unitary Shimura variety as in Proposition 2.27(3), or more precisely [TX13⁺b, Conjecture 5.18] holds true. Then the same argument above can generalize the Theorem in complete generality to the case when p is only assumed to be unramified, an every prime ideal \mathfrak{p}_i behaves “in an independent way”. More precisely, we fix $r_i \leq \frac{d_i}{2}$ for all i , where $d_i = \#(S_\infty^c \cap \Sigma_{\mathfrak{p}_i})$ and $\Sigma_{\mathfrak{p}_i}$ is the subset of p -adic embeddings that induce the prime \mathfrak{p}_i . Then the Goren–Oort cycles would be parameterized by h -tuples whose i th component is a semi-meander with d_i nodes and r_i arcs. Under the genericity condition: the eigenvalues of $\rho_\pi(\text{Frob}_{\mathfrak{p}_i})$ avoid certain roots of unity, the cohomology of the Goren–Oort cycles generate the subspace of the cohomology $H_{\text{et}}^d(\text{Sh}_{K_p}(G_{\mathbf{S}, \mathbf{T}})_{\overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{S}, \mathbf{T}}^{(k, w)})[\pi]$ where certain analogs of $\mathfrak{F}_{\mathfrak{p}_i}''$ acts with appropriate eigenvalues determined by r_i and $\text{Frob}_{\mathfrak{p}_i}$.

¹⁹Note that we still need α_π/β_π to avoid certain roots of unity to get (4.4.5).

Without [TX13⁺b, Conjecture 5.18], we can only prove the analogous statement when $r_i = \frac{d_i}{2}$ for i , that is, in the case for Tate cycles.²⁰ Moreover, since we can not distinguish the actions of each $\mathfrak{F}_{\mathfrak{p}_i}''$, we would have to assume that the eigenvalues of $\rho_\pi(\text{Frob}_{\mathfrak{p}_i})$ are “generic”, so that all eigenvalues of Frob_p^g acting on $H_{\text{et}}^2(\text{Sh}_{K_p}(G_{\mathbf{S},\mathbf{T}})_{\overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{S},\mathbf{T}}^{(k,w)})$ are “as distinct as possible”, where g stands for the least common multiple of the inertia degrees of the \mathfrak{p}_i s. For example, this excludes the case when both \mathfrak{p}_1 and \mathfrak{p}_2 have inertia degree 2 and $\text{Frob}_{\mathfrak{p}_1}$ and $\text{Frob}_{\mathfrak{p}_2}$ have the same set of eigenvalues (which would be okay if [TX13⁺b, Conjecture 5.18] is known).

- (2) It would be interesting to know, when p is ramified in F/\mathbb{Q} , whether one can prove a similar result for the special fiber of the splitting model of the Hilbert modular variety of Pappas and Rapoport. The construction of the corresponding Goren–Oort divisors is discussed in [RX14⁺].
- (3) The construction of these Goren–Oort cycles a priori depends on the CM extension E of F . This is also responsible for avoiding $2n$ th roots of unity as opposed to just n th roots of unity. We think these two issues are purely technical, as our current technique relies very much on the PEL moduli interpretation.
- (4) In the case of $r = d/2$ (namely the case for Tate classes), the map (4.4.1) is injective as long as $\alpha_\pi \neq \beta_\pi$. We need α_π/β_π to avoid more roots of unities so that both sides of (4.4.1) have the same dimension.
- (5) It is attempting to ask the question: to what extend does this imply the semisimplicity of the Frobenius action? Unfortunately, our theorem is in its strongest form only when $\alpha_\pi \neq \beta_\pi$, where $\rho_\pi(\text{Frob}_{p^g})$ is automatically semi-simple. So one cannot say too much about ρ_π . However, ρ_π as a representation of $\text{Gal}_{\mathbb{Q}}$ might be reducible (e.g. when π is CM). In this case, our theorem might provide some insight into the semi-simplicity of $H_{\text{et}}^d(X_{\overline{\mathbb{F}}_p}, \mathbb{Q}_\ell)[\pi]$ as a representation of $\text{Gal}_{\mathbb{Q}}$. See also [Ne15⁺].
- (6) It is also attempting to ask: in the case of $r = d/2$ (the Tate classes case), is the determinant of the intersection matrix related to the higher derivatives of the local L-function (to get certain local version of the Beilinson–Bloch Conjecture)? We feel the answer might be negative. Note that the intersection matrix is always a power of $(\alpha_\pi - \beta_\pi)$, but the higher derivatives of the local L-functions can involve factors of the form $\alpha^s - \beta^s$ for $s < d/2$. We also point out that, in the recent preprint [YZ15⁺] of Z. Yun and W. Zhang, they seem to suggest a new philosophy for higher derivatives of *global* L-functions. We do not know how to compare the determinant of our intersection matrix to their formulation.

Remark 4.6. It is a very interesting question to understand the case when $\alpha_\pi = \beta_\pi$. We explain this in the quadratic case. Let F be a real quadratic field in which p is inert. Let $\pi \in \mathcal{A}_{(2,2)}$ be an automorphic representation with trivial central character, defined over \mathbb{R} . Suppose that π appears in the cohomology of the quaternionic Shimura variety $X = \text{Sh}_K(G_{\{v_1, v_2\}, \emptyset})$, where v_1 and v_2 are two finite prime-to- p places of F (so that X is proper for simplicity). Suppose that the unramified $\alpha_\pi = \beta_\pi = \pm p$. For instance, when π comes from the base change of a usual modular form corresponding to an elliptic curve over \mathbb{Q} which has supersingular (good) reduction at p , then the local Satake parameters of π at p are $\alpha_\pi = \beta_\pi = p$.

In this case, $H_{\text{et}}^2(X_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(1))[\pi]$ is 4-dimensional on which Frob_{p^2} acts trivially. More precisely, as pointed out by Prasanna, the action of Frob_p on this four dimensional subspace has two eigenvalues: α_π/p (with multiplicity 3) and $-\alpha_\pi/p$ (with multiplicity 1). There are two Goren–Oort cycles, both given by a collection of \mathbb{P}^1 's. The π -isotypical components of their cycle classes contribute

²⁰If $r_i < \frac{d_i}{2}$ for some i , the determinant of the intersection matrix would involve the knowledge of different powers of the action of $\mathfrak{F}_{\mathfrak{p}_i}''$. But we only have the information of their product $S_p^{-1} \cdot F^2 := \prod_{i=1}^h \mathfrak{F}_{\mathfrak{p}_i}''$. On the other hand, the case $r_i = \frac{d_i}{2}$ is fine, because we only use the Hecke operators, whose action on the cohomology is known.

non-trivially to the subspace with Frob_p -eigenvalue $-\alpha_\pi/p$. We claim that the π -isotypical component of the their cycles classes does not contribute to the subspace with Frob_p -eigenvalue α_π/p . Indeed, the intersection matrix B given above is degenerate (having rank 1). Note that, for any cuspidal π , the π -isotypical component of the rational Néron–Severi group of X is orthogonal to the subspace of ample line bundles, and Hodge index theorem implies that the intersection pairing on the π -isotypical component is non-degenerate. So the degeneracy of the intersection matrix means that the contribution from the Goren–Oort cycles is indeed a one-dimensional subspace of $H_{\text{et}}^2(X_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(1))[\pi]$, namely the subspace with Frob_p -eigenvalue $-\alpha_\pi/p$.

We think this phenomenon is comparable to the case of Heegner points: when the rank of the elliptic curve is one, the Heegner point gives a canonical generator of the Mordell–Weil group tensored with \mathbb{Q} ; however, when the rank of the elliptic curve is strictly bigger than one (“generic dimension”), the Heegner point becomes torsion. In our case, the classes of the Goren–Oort cycles are similar to Heegner points. When the dimension of the corresponding Frobenius (generalized) eigenspace is “generic”, the classes of the Goren–Oort cycles give a canonical basis, but when the dimension is bigger than the generic, the contribution from the Goren–Oort cycles tends to degenerate.

5. COMPUTATION OF THE INTERSECTION MATRIX

The aim of this section is to establish Theorem 4.3 and hence to finish the proof of the main theorems. We keep the notation from the previous section.

Notation 5.1. To simplify notation, we suppress the automorphic sheaf $\mathcal{L}_{\mathbf{s}, \mathbf{T}}^{(k, w)}$, the level structure K_p , the change of base to $\overline{\mathbb{F}}_p$, and the subscript et from the notation of cohomology groups, as they are all fixed throughout this section. For example, we write

$$H^\star(\text{Sh}(G_{\mathbf{s}, \mathbf{T}})_a)(r) \text{ for } H_{\text{et}}^\star(\text{Sh}_{K_p}(G_{\mathbf{s}, \mathbf{T}})_{a, \overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{s}, \mathbf{T}}^{(k, w)}(r)|_{\text{Sh}_{K_p}(G_{\mathbf{s}, \mathbf{T}})_a}).$$

This should not cause any confusion because all the automorphic sheaves are compatible on the Goren–Oort cycles. As in Theorem 4.3, we fix a choice of system of shifts \mathbf{t}_a of the correspondences $\text{Sh}_{K_p}(G_{\mathbf{s}_a, \mathbf{T}_a}) \xleftarrow{\pi_a} \text{Sh}_{K_p}(G_{\mathbf{s}, \mathbf{T}})_a \hookrightarrow \text{Sh}_{K_p}(G_{\mathbf{s}, \mathbf{T}})$ as in Subsection 3.7.

Before going into the intricate induction, we first handle a few simple but essential cases. The general case will be essentially reduced to these cases.

5.2. The case of $r = 1$ and $\mathbf{a} = \mathbf{b}$. This is the case where the corresponding periodic semi-meanders are given as

$$\mathbf{a} = \mathbf{b} = \begin{array}{ccccccc} & & \uparrow & & \text{---} & \text{---} & \uparrow \\ \cdots & + & \bullet & + & \cdots & + & \bullet & + & \cdots & + & \bullet & + & \cdots \\ & & \tau^- & & \text{---} & & \tau & & \end{array}$$

(or their shifts), linking τ with $\tau^- = \sigma^{-n_\tau} \tau$.

Unwinding the definition, we have the following commutative diagram

$$\begin{array}{ccc} H^{d-2}(\text{Sh}(G_{\mathbf{s}_a, \mathbf{T}_a})) & \xrightarrow{\text{Res}_a \circ \text{Gys}_a} & H^{d-2}(\text{Sh}(G_{\mathbf{s}_a, \mathbf{T}_a})) \\ \downarrow \pi_a^* & & \uparrow \pi_{a,!} \\ H^{d-2}(\text{Sh}(G_{\mathbf{s}, \mathbf{T}})_a) & \xrightarrow{\text{Gysin}} H^d(\text{Sh}(G_{\mathbf{s}, \mathbf{T}}))(1) \xrightarrow{\text{Restr.}} & H^d(\text{Sh}(G_{\mathbf{s}, \mathbf{T}})_a)(1). \end{array}$$

Recall that $\text{Sh}(G_{\mathbf{s}, \mathbf{T}})_a$ is a \mathbb{P}^1 -bundle over $\text{Sh}(G_{\mathbf{s}_a, \mathbf{T}_a})$, hence π_a^* and $\pi_{a,!}$ are both isomorphisms. By the excessive intersection formula [Fu98, §6.3], the composition of the bottom line is given by the cup product with the first Chern class of the normal bundle of the embedding $\text{Sh}(G_{\mathbf{s}, \mathbf{T}})_a \hookrightarrow \text{Sh}(G_{\mathbf{s}, \mathbf{T}})$, which is isomorphic to $-2p^{n_\tau}$ times the canonical quotient ample line bundle for the \mathbb{P}^1 -bundle

given by π_a , according to Proposition 2.31(2). Therefore, the top line morphism $\text{Res}_a \circ \text{Gys}_a$ is nothing but multiplication by $-2p^{n_\tau} = -2p^{\ell(a)}$.

5.3. The case of $d = 2$ and $r = 1$ with $a \neq b$. This is the case where the corresponding periodic semi-meanders are given as

$$a = \cdots + \underset{\tau^-}{\bullet} + \cdots + \underset{\tau}{\bullet} + \cdots \quad \text{and} \quad b = \cdots + \underset{\tau^-}{\bullet} + \cdots + \cdots + \underset{\tau}{\bullet} + \cdots$$

(or their simultaneous shifts). Let τ^- denote the left end-node of the arc of a , and τ is the right end-node. We have $\tau^+ = \tau^-$. Here the meaning of left and right refers to the xy -plane presentation of a , as explained in 3.1.

Unwinding the definition, the morphism $\text{Res}_a \circ \text{Gys}_b$ is the composition of the following commutative diagram from the upper-left to the lower-right (first rightward and then downward):

$$\begin{array}{ccccc} H^0(\text{Sh}(G_{S_b, T_b})) & \xrightarrow{\pi_b^*} & H^0(\text{Sh}(G_{S, T})_{\tau^-}) & \xrightarrow{\text{Gysin}} & H^2(\text{Sh}(G_{S, T}))(1) \\ & \searrow & \downarrow \text{Restr.} & & \downarrow \text{Restr.} \\ & & H^0(\text{Sh}(G_{S, T})_{\{\tau^-, \tau\}}) & \xrightarrow{\text{Gysin}} & H^2(\text{Sh}(G_{S, T})_\tau)(1) \\ & & & \searrow \text{Tr}_{\pi_a} & \downarrow \pi_{a,!} \\ & & & & H^0(\text{Sh}(G_{S_a, T_a})). \end{array}$$

Here, the commutativity of the square follows from the fact that the corresponding morphisms of Shimura varieties form a Cartesian square (as a and b take the vanishing locus of two different partial Hasse-invariants), and Tr_{π_a} is the trace map induced by the finite étale map (of zero-dimensional Shimura varieties)

$$\text{Sh}(G_{S, T})_{\{\tau^-, \tau\}} \hookrightarrow \text{Sh}(G_{S, T})_\tau \xrightarrow{\pi_a} \text{Sh}(G_{S_a, T_a}),$$

and the natural isomorphism between the pullback of the automorphic sheaf on $\text{Sh}(G_{S_a, T_a})$ with that on $\text{Sh}(G_{S, T})_\tau$.

By Theorem 2.32(3), the diagonal composition from upper-left to lower-right, or equivalently the morphism $\text{Res}_a \circ \text{Gys}_b$, is $T_p \circ (\eta_{S_a, S_b, (n)}^*)^{-1}$, where $\eta_{S_a, S_b, (n)}^*$ is the link morphism associated to the trivial link $\eta_{S_b, S_a} : S_b \rightarrow S_a$ with indentation degree $n = 2n_{\tau^-} = -(\ell(a) - \ell(b) - g)$ and shift $\varpi_{\bar{q}} t_a^{-1} t_b$. Thus the inverse $(\eta_{S_b, S_a, (n)}^*)^{-1} = (\eta_{S_a, S_b, (-n)}^{-1})^*$ is the link morphism associated to the link η^{-1} with indentation degree $\ell(a) - \ell(b) - g$ and shift $\varpi_{\bar{q}}^{-1} t_a t_b^{-1}$. This proves Theorem 4.3(3) for the given case.

5.4. The case of $r = 1$, $d > 2$, and $\langle a, b \rangle = v^{m_v}$. Assume that $m_v > 0$ first. In this situation, the corresponding periodic semi-meanders, up to shifting, are given by

$$(5.4.1) \quad \begin{array}{l} a = \cdots + \underset{\tau^-}{\bullet} + \cdots + \underset{\tau}{\bullet} + \cdots + \underset{\tau^+}{\bullet} + \cdots \\ b = \cdots + \underset{\tau^-}{\bullet} + \cdots + \underset{\tau}{\bullet} + \cdots + \underset{\tau^+}{\bullet} + \cdots \end{array} \quad \text{and}$$

Note that the two arcs in a and b must be adjacent, otherwise, $\langle a, b \rangle = 0$. Let τ denote the left end-node of the (unique) arc in a , as shown in the pictures above. Then τ^- is the left end-node of the arc in b , and τ^+ is the right end-node of the arc in a . So if $\tau = \sigma^{-n_\tau} \tau^+$ and $\tau^- = \sigma^{-n_\tau} \tau$, then $m_v = m_v(a, b) = n_\tau + n_{\tau^+}$.

Unwinding the definition, the morphism $\text{Res}_a \circ \text{Gys}_b$ is the composition of the following commutative diagram from the upper-left to the lower-right:

$$(5.4.2) \quad \begin{array}{ccccc} H^{d-2}(\text{Sh}(G_{S_b, T_b})) & \xrightarrow{\pi_b^*} & H^{d-2}(\text{Sh}(G_{S, T})_\tau) & \xrightarrow{\text{Gysin}} & H^d(\text{Sh}(G_{S, T}))(1) \\ & \searrow \pi_b^* & \downarrow \text{Restr.} & & \downarrow \text{Restr.} \\ & & H^{d-2}(\text{Sh}(G_{S, T})_{\{\tau, \tau^+\}}) & \xrightarrow{\text{Gysin}} & H^d(\text{Sh}(G_{S, T})_{\tau^+})(1) \\ & & & \searrow (\theta^{-1})^*, \cong & \downarrow \pi_{a,!} \\ & & & & H^{d-2}(\text{Sh}(G_{S_a, T_a})). \end{array}$$

By Theorem 2.32(2), the morphism

$$\theta : \text{Sh}(G_{S, T})_{\{\tau, \tau^+\}} \hookrightarrow \text{Sh}(G_{S, T})_{\tau^+} \xrightarrow{\pi_a} \text{Sh}(G_{S_a, T_a})$$

is an isomorphism, and the composition

$$\text{Sh}(G_{S_a, T_a}) \xleftarrow{\theta^{-1}} \text{Sh}(G_{S, T})_{\{\tau, \tau^+\}} \hookrightarrow \text{Sh}(G_{S, T})_\tau \xrightarrow{\pi_b} \text{Sh}(G_{S_b, T_b})$$

is exactly the link morphism

$$\eta_{a, b, (z), \#} : \text{Sh}(G_{S_a, T_a}) \longrightarrow \text{Sh}(G_{S_b, T_b}),$$

associated to the link $\eta_{a, b} : S_a \rightarrow S_b$ given by

$$(5.4.3) \quad \begin{array}{ccccccc} \cdots + & \bullet & + \cdots + & \tau^- & + \cdots + & \tau & + \cdots + & \tau^+ & + \cdots + & \bullet & + \cdots \\ & \downarrow & & \searrow & & & & & & \downarrow & \\ \cdots + & \bullet & + \cdots + & & + \cdots + & & + \cdots + & & + \cdots + & \bullet & + \cdots \end{array},$$

with shift $t_a t_b^{-1}$ and indentation degree z equal to $\ell(a) - \ell(b)$ if \mathfrak{p} splits in E/F and to 0 if \mathfrak{p} is inert in E/F . Therefore, $\text{Res}_a \circ \text{Gys}_b$ is exactly $p^{v(\eta_{a, b})/2} \eta_{a, b, (z)}^* = p^{m_v/2} \eta_{a, b, (z)}^*$ (note the normalization in (2.25.1)), verifying Theorem 4.3(2).

We now come to the case where m_v is negative. In this case, the picture of a and b in (5.4.1) are swapped. Then we have a commutative diagram similar to (5.4.2):

$$\begin{array}{ccccc} H^{d-2}(\text{Sh}(G_{S_b, T_b})) & \xrightarrow{\pi_b^*} & H^{d-2}(\text{Sh}(G_{S, T})_{\tau^+}) & \xrightarrow{\text{Gysin}} & H^d(\text{Sh}(G_{S, T}))(1) \\ & \searrow \cong & \downarrow \text{Restr.} & & \downarrow \text{Restr.} \\ & & H^{d-2}(\text{Sh}(G_{S, T})_{\{\tau, \tau^+\}}) & \xrightarrow{\text{Gysin}} & H^d(\text{Sh}(G_{S, T})_\tau)(1) \\ & & & \searrow & \downarrow \pi_{a,!} \\ & & & & H^{d-2}(\text{Sh}(G_{S_a, T_a})), \end{array}$$

and the composed diagonal morphism gives $\text{Res}_a \circ \text{Gys}_b$. Let $\eta_{b, a} : S_b \rightarrow S_a$ denote the inverse link of $\eta_{a, b}$. Since a and b are obtained by swapping with each other from the previous case, the link morphism $\eta_{b, a, (-z), \#} : \text{Sh}(G_{S_b, T_b}) \rightarrow \text{Sh}(G_{S_a, T_a})$ with shift $t_b t_a^{-1}$ exists, where $z = \ell(a) - \ell(b)$ if \mathfrak{p} splits in E and $z = 0$ if \mathfrak{p} is inert in E . Note also that $\eta_{b, a, (-z), \#}$ is finite flat of degree $p^{-m_v} = p^{v(\eta_{b, a})}$ by Theorem 2.32. One sees easily that $\text{Res}_a \circ \text{Gys}_b = \text{Tr}_{\eta_{b, a, (-z), \#}}$. By Lemma 2.29(3), this is exactly $p^{-m_v/2} (\eta_{b, a, (-z)}^*)^{-1} = p^{(\ell(a) + \ell(b))/2} \eta_{a, b, (z)}^*$. This proves Theorem 4.3(2) in this case.

5.5. Decomposition of periodic semi-meanders. Before proceeding to the inductive proof, we discuss certain ways to “decompose” periodic semi-meanders appearing in the induction. Let $\mathbf{a} \in \mathfrak{B}_S^r$ be a periodic semi-meander. We call a subset Δ of r' arcs ($r' \leq r$) in \mathbf{a} *saturated*, if for each arc δ belonging to Δ , any arc that lies below δ in the sense of Notation 3.6 belongs to

Δ . For example, if $\mathbf{a} = \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet$, the subset $\Delta = \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet^{21}$ is saturated, but $\bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet$ is not.

Now fix a saturated Δ . We use \mathbf{a}^b to denote the periodic semi-meander for S given by all the arcs in Δ and then adjoining semi-lines to the rest of nodes. Then $S_{\mathbf{a}^b}$ is the union of S and all nodes connected to an arc in Δ . We use $\mathbf{a}_{\text{res}} = \mathbf{a} \setminus \Delta$ to denote the periodic semi-meander for $S_{\mathbf{a}^b}$ obtained by removing all the arcs in Δ and replacing their end-nodes by plus signs. In the example

above, $\mathbf{a}^b = \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet$ and $\mathbf{a}_{\text{res}} = \bullet \bullet + + + + + \bullet \bullet$, where the plus signs indicates points corresponding to $S_{\mathbf{a}^b, \infty}$.

By the construction of the Goren–Oort cycles, we have the following commutative diagram, where the middle square is Cartesian.

$$(5.5.1) \quad \begin{array}{ccccc} & \text{Sh}(G_{S,T})_{\mathbf{a}} & \hookrightarrow & \text{Sh}(G_{S,T})_{\mathbf{a}^b} & \hookrightarrow & \text{Sh}(G_{S,T}) \\ & \downarrow & & \downarrow \pi_{\mathbf{a}^b} & & \\ \pi_{\mathbf{a}} \swarrow & \text{Sh}(G_{S_{\mathbf{a}^b}, T_{\mathbf{a}^b}})_{\mathbf{a}_{\text{res}}} & \hookrightarrow & \text{Sh}(G_{S_{\mathbf{a}^b}, T_{\mathbf{a}^b}}) & & \\ & \downarrow \pi_{\mathbf{a}_{\text{res}}} & & & & \\ & \text{Sh}(G_{S_{\mathbf{a}}, T_{\mathbf{a}}}) & & & & \end{array}$$

Since the construction of this diagram comes from the unitary Shimura varieties, we point out that, as explained near (3.7.3), the shift of the correspondence $\text{Sh}(G_{S_{\mathbf{a}}, T_{\mathbf{a}}}) \xleftarrow{\pi_{\mathbf{a}_{\text{res}}}} \text{Sh}(G_{S_{\mathbf{a}^b}, T_{\mathbf{a}^b}})_{\mathbf{a}_{\text{res}}} \hookrightarrow \text{Sh}(G_{S_{\mathbf{a}^b}, T_{\mathbf{a}^b}})$ is $\mathbf{t}_{\mathbf{a}^b, \mathbf{a}} = \mathbf{t}_{\mathbf{a}} \mathbf{t}_{\mathbf{a}^b}^{-1}$. From the commutative diagram, we can decompose the morphisms $\text{Res}_{\mathbf{a}}$ and $\text{Gys}_{\mathbf{a}}$ as follows:

$$\begin{aligned} \text{Gys}_{\mathbf{a}} : H^{d-2r}(\text{Sh}(G_{S_{\mathbf{a}}, T_{\mathbf{a}}})) &\xrightarrow{\pi_{\mathbf{a}_{\text{res}}}^*} H^{d-2r}(\text{Sh}(G_{S_{\mathbf{a}^b}, T_{\mathbf{a}^b}})_{\mathbf{a}_{\text{res}}}) \xrightarrow{\text{Gysin}} H^{d-2r'}(\text{Sh}(G_{S_{\mathbf{a}^b}, T_{\mathbf{a}^b}}))(r-r') \\ &\xrightarrow{\pi_{\mathbf{a}^b}^*} H^{d-2r'}(\text{Sh}(G_{S,T})_{\mathbf{a}^b})(r-r') \xrightarrow{\text{Gysin}} H^d(\text{Sh}(G_{S,T}))(r) \end{aligned}$$

and

$$\begin{aligned} \text{Res}_{\mathbf{a}} : H^d(\text{Sh}(G_{S,T}))(r) &\xrightarrow{\text{Restr.}} H^d(\text{Sh}(G_{S,T})_{\mathbf{a}^b})(r) \xrightarrow{\pi_{\mathbf{a}^b, !}} H^{d-2r'}(\text{Sh}(G_{S_{\mathbf{a}^b}, T_{\mathbf{a}^b}}))(r-r') \\ &\xrightarrow{\text{Restr.}} H^{d-2r'}(\text{Sh}(G_{S_{\mathbf{a}^b}, T_{\mathbf{a}^b}})_{\mathbf{a}_{\text{res}}})(r-r') \xrightarrow{\pi_{\mathbf{a}_{\text{res}}, !}} H^{d-2r}(\text{Sh}(G_{S_{\mathbf{a}}, T_{\mathbf{a}}})). \end{aligned}$$

Here, to get the decomposition for $\text{Res}_{\mathbf{a}}$, we have used the fact that the trace map $\text{Tr}_{\pi_{\mathbf{a}}}$ can be factorized as

$$R^{2r} \pi_{\mathbf{a}*} \overline{\mathbb{Q}}_{\ell}(r) \cong R^{2r-2r'} \pi_{\mathbf{a}_{\text{res}}*} (R^{2r'} \pi_{\mathbf{a}^b*} \overline{\mathbb{Q}}_{\ell})(r) \xrightarrow{\text{Tr}_{\pi_{\mathbf{a}^b}}} R^{2r-2r'} \pi_{\mathbf{a}_{\text{res}}*} (\overline{\mathbb{Q}}_{\ell})(r-r') \xrightarrow{\text{Tr}_{\pi_{\mathbf{a}_{\text{res}}}}} \overline{\mathbb{Q}}_{\ell}.$$

Summing up everything in short, we obtain thus

$$\text{Gys}_{\mathbf{a}} = \text{Gys}_{\mathbf{a}^b} \circ \text{Gys}_{\mathbf{a}_{\text{res}}}, \quad \text{Res}_{\mathbf{a}} = \text{Res}_{\mathbf{a}_{\text{res}}} \circ \text{Res}_{\mathbf{a}^b}, \quad \text{and } \mathbf{t}_{\mathbf{a}} = \mathbf{t}_{\mathbf{a}^b} \mathbf{t}_{\mathbf{a}^b, \mathbf{a}}.$$

We will apply this to appropriate Δ 's to reduce the calculation to $\text{Sh}(G_{S_{\mathbf{a}^b}, T_{\mathbf{a}^b}})$ and reduce the inductive proof essentially to the cases considered above.

²¹Here Δ is only the set of the arcs, not including the nodes in the picture.

5.6. Decomposition of periodic semi-meanders continued. We will also encounter the following situation: assume that the set of arcs in a periodic semi-meander \mathbf{a} is the disjoint union of two *saturated* subsets Δ and Δ' . Put $s = \#\Delta$ and $s' = \#\Delta'$ so that $r = s + s'$. We will show that Δ and Δ' “behave” independently.

We write \mathbf{a}^b (resp. $\mathbf{a}^{b'}$) for the periodic semi-meander for \mathbf{S} given by all arcs in Δ (resp. Δ') and then adjoining semi-lines to the rest of the nodes. We put \mathbf{a}_{res} (resp. \mathbf{a}'_{res}) for the periodic semi-meander for $\mathbf{S}_{\mathbf{a}^b}$ (resp. $\mathbf{S}_{\mathbf{a}^{b'}}$) obtained by removing all arcs in Δ (resp. Δ') and replacing all their end-nodes by plus signs.

In this case, in view of the construction of the Goren–Oort cycle $\text{Sh}(G_{\mathbf{S},\mathbf{T}})_{\mathbf{a}}$, we could either go through the arcs in Δ first, or the arcs in Δ' first. So we have the following commutative *Cartesian* diagram:

$$(5.6.1) \quad \begin{array}{ccccc} \text{Sh}(G_{\mathbf{S},\mathbf{T}}) & \longleftarrow & \text{Sh}(G_{\mathbf{S},\mathbf{T}})_{\mathbf{a}^b} & \xrightarrow{\pi_{\mathbf{a}^b}} & \text{Sh}(G_{\mathbf{S}_{\mathbf{a}^b},\mathbf{T}_{\mathbf{a}^b}}) \\ \uparrow & & \uparrow & & \uparrow \\ \text{Sh}(G_{\mathbf{S},\mathbf{T}})_{\mathbf{a}^{b'}} & \longleftarrow & \text{Sh}(G_{\mathbf{S},\mathbf{T}})_{\mathbf{a}} & \xrightarrow{\pi_{\Delta}} & \text{Sh}(G_{\mathbf{S}_{\mathbf{a}^b},\mathbf{T}_{\mathbf{a}^b}})_{\mathbf{a}_{\text{res}}} \\ \downarrow \pi_{\mathbf{a}^{b'}} & & \downarrow \pi_{\Delta'} & & \downarrow \pi_{\mathbf{a}_{\text{res}}} \\ \text{Sh}(G_{\mathbf{S}_{\mathbf{a}^{b'}},\mathbf{T}_{\mathbf{a}^{b'}}}) & \longleftarrow & \text{Sh}(G_{\mathbf{S}_{\mathbf{a}^{b'}},\mathbf{T}_{\mathbf{a}^{b'}}})_{\mathbf{a}'_{\text{res}}} & \xrightarrow{\pi_{\mathbf{a}'_{\text{res}}}} & \text{Sh}(G_{\mathbf{S}_{\mathbf{a}},\mathbf{T}_{\mathbf{a}}}), \end{array}$$

where π_{Δ} and $\pi_{\Delta'}$ are the morphisms defined by the natural pull-back of upper-right and lower-left Cartesian squares, respectively. By Remark 2.16 the shifts satisfies the following equality

$$(5.6.2) \quad t_{\mathbf{a}^b} t_{\mathbf{a}^b, \mathbf{a}} = t_{\mathbf{a}} = t_{\mathbf{a}^{b'}} t_{\mathbf{a}^{b'}, \mathbf{a}} \quad \text{in } E^{\times, \text{cl}} \backslash \mathbb{A}_E^{\infty, \times} / \mathcal{O}_{E_p}^{\times}.$$

This implies that both π_{Δ} and $\pi_{\Delta'}$ are iterated \mathbb{P}^1 -bundles of relative dimensions s and s' respectively. We use $\pi_{\Delta,!}$ to denote the natural morphism

$$\begin{aligned} \pi_{\Delta,!} : H_{\text{et}}^{\star}(\text{Sh}_{K_p}(G_{\mathbf{S},\mathbf{T}})_{\mathbf{a}, \overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{S},\mathbf{T}}^{(k,w)}(s)) &\xrightarrow{\cong} H_{\text{et}}^{\star}(\text{Sh}_{K_p}(G_{\mathbf{S}_{\mathbf{a}^b},\mathbf{T}_{\mathbf{a}^b}})_{\mathbf{a}_{\text{res}}, \overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{S}_{\mathbf{a}^b},\mathbf{T}_{\mathbf{a}^b}}^{(k,w)} \otimes R\pi_{\Delta*} \overline{\mathbb{Q}}_{\ell}(s)) \\ &\xrightarrow{\text{Tr } \pi_{\Delta}} H_{\text{et}}^{\star-2s}(\text{Sh}_{K_p}(G_{\mathbf{S}_{\mathbf{a}^b},\mathbf{T}_{\mathbf{a}^b}})_{\mathbf{a}_{\text{res}}, \overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{S}_{\mathbf{a}^b},\mathbf{T}_{\mathbf{a}^b}}^{(k,w)}), \end{aligned}$$

where the last map is induced by the trace isomorphism $R^{2s}\pi_{\Delta,*}(\overline{\mathbb{Q}}_{\ell}(s)) \cong \overline{\mathbb{Q}}_{\ell}$.

As a consequence of the Cartesian property and Theorem 2.32(1), we have the following commutative diagram (which is placed into (5.6.1) vertically on the right)

$$(5.6.3) \quad \begin{array}{ccccc} H^{d-2s'}(\text{Sh}(G_{\mathbf{S}_{\mathbf{a}^{b'}},\mathbf{T}_{\mathbf{a}^{b'}}})_{\mathbf{a}'_{\text{res}}})(s) & \xrightarrow{\pi_{\Delta'}^*} & H^{d-2s'}(\text{Sh}(G_{\mathbf{S},\mathbf{T}})_{\mathbf{a}})(s) & \xrightarrow{\text{Gysin}} & H^d(\text{Sh}(G_{\mathbf{S},\mathbf{T}})_{\mathbf{a}^b})(s+s') \\ \downarrow \pi_{\mathbf{a}'_{\text{res}},!} & & \downarrow \pi_{\Delta,!} & & \downarrow \pi_{\mathbf{a}^b,!} \\ H^{d-2s-2s'}(\text{Sh}(G_{\mathbf{S}_{\mathbf{a}},\mathbf{T}_{\mathbf{a}}})) & \xrightarrow{\pi_{\mathbf{a}_{\text{res}}}^*} & H^{d-2s-2s'}(\text{Sh}(G_{\mathbf{S}_{\mathbf{a}^b},\mathbf{T}_{\mathbf{a}^b}})_{\mathbf{a}_{\text{res}}}) & \xrightarrow{\text{Gysin}} & H^{d-2s}(\text{Sh}(G_{\mathbf{S}_{\mathbf{a}^b},\mathbf{T}_{\mathbf{a}^b}}))(s'). \end{array}$$

5.7. Inductive proof of Theorem 4.3. We now start the proof of Theorem 4.3 by induction on $d = \#\mathbf{S}_{\infty}^c$ or equivalently the dimension of the Shimura variety $\text{Sh}(G_{\mathbf{S},\mathbf{T}})$ (and also on r by keeping $d - 2r$ fixed throughout the induction). The base case $d = 0$ and $d = 1$ are trivial (as there is no nontrivial periodic semi-meander).

We now assume that Theorem 4.3 holds for all Shimura varieties $\text{Sh}_K(G_{\mathbf{S},\mathbf{T}})$ with $\#\mathbf{S}_{\infty}^c < d$. We now fix \mathbf{S}, \mathbf{T} so that $\#\mathbf{S}_{\infty}^c = d$. The case of $r = 0$ is clear. We henceforth assume that $r > 0$.

Let $\mathbf{a}, \mathbf{b} \in \mathfrak{B}_{\mathbf{S}}^c$ be as in Theorem 4.3. We fix a *basic* arc $\delta_{\mathbf{b}}$ of \mathbf{b} , with right end-node $\tau \in \mathbf{S}_{\infty}^c$ (and left end-node $\tau^- \in \mathbf{S}_{\infty}^c$). As in Subsection 5.5, we use $\mathbf{b}_{\text{res}} \in \mathfrak{B}_{\mathbf{S} \cup \{\tau, \tau^-\}}^{r-1}$ to denote the periodic semi-meander $\mathbf{b} \setminus \delta_{\mathbf{b}}$ obtained by removing $\delta_{\mathbf{b}}$ from \mathbf{b} and replacing its end-nodes by plus signs. We

will use δ_b itself to denote the corresponding \mathfrak{b}^b , that is, we also view δ_b as a periodic semi-meander for \mathbf{S} with only one arc δ_b (and $d - 2$ semi-lines).

The basic idea is to factor the Gysin map Gys_b using δ_b , in the sense of Subsection 5.5, and to factor the restriction map Res_a according to the following list of four cases.

- (i) The two nodes τ, τ^- are both linked to semi-lines in \mathfrak{a} . This forces us to fall into the case (1) of Theorem 4.3.
- (ii) There is a (basic) arc δ_a in \mathfrak{a} linking τ^- to τ from left to right, so that δ_a and δ_b form a contractible loop in $D(\mathfrak{a}, \mathfrak{b})$. In other words, δ_a and δ_b are the same (up to deformation of the arcs). We shall reduce the proof of Theorem 4.3 to the case for $\mathbf{S}' = \mathbf{S} \cup \{\tau, \tau^-\}$, $\mathbf{T}' = \mathbf{T} \cup \{\tau\}$, $\mathfrak{a}_{\text{res}} = \mathfrak{a} \setminus \delta_a$ and $\mathfrak{b}_{\text{res}}$ and it hence follows from the inductive hypothesis. In particular, we shall see that the contractible loop δ_a and δ_b contributes a factor of $-2p^{\ell(\delta_b)}$.
- (iii) There is an arc δ_a in \mathfrak{a} connecting τ to τ^- wrapped around the cylinder from right to left. In other words, δ_a and δ_b together form a non-contractible loop in $D(\mathfrak{a}, \mathfrak{b})$. This can only happen if $r = d/2$. We will show that the composition $\text{Res}_a \circ \text{Gys}_b$ is essentially the T_p -operator composed with $\text{Res}_{\mathfrak{a} \setminus \delta_a} \circ \text{Gys}_{\mathfrak{b} \setminus \delta_b}$ for the Shimura variety with $\mathbf{S}' = \mathbf{S} \cup \{\tau, \tau^-\}$ and $\mathbf{T}' = \mathbf{T} \cup \{\tau\}$, up to some link morphism which we make explicit later.
- (iv) Neither of above happens. Then, in \mathfrak{a} , either τ is connected by an arc whose other end-node is not τ^- , and/or τ^- is connected by an arc whose other end-node is not τ . In either case, we will reduce to a case with the two nodes τ and τ^- removed, after composing with a certain link morphism.

We now treat each of the cases separately.

5.8. Case (i). This is the case when τ and τ^- are connected to semi-lines in \mathfrak{a} . This implies that $\langle \mathfrak{a} | \mathfrak{b} \rangle = 0$. So we are in the situation of Theorem 4.3(1). We need to show that the π -isotypical component of $\text{Res}_a \circ \text{Gys}_b$ factors through the cohomology of a Shimura variety of smaller dimension. Let \mathfrak{a}^* denote the periodic semi-meander for \mathbf{S} given by removing the two semi-lines of \mathfrak{a} connected to τ and τ^- and reconnecting τ and τ^- by a (basic) arc. Note that this is possible because δ_b is a basic arc, so τ and τ^- are adjacent nodes in the band for \mathbf{S} . In particular, $\mathfrak{a}^* \in \mathfrak{B}_S^{r+1}$.

By the discussion of Subsection 5.5, we see that the morphism $\text{Res}_a \circ \text{Gys}_b$ is the composition from top-left to the bottom-right of the following commutative diagram by going first downwards and then rightwards.

$$\begin{array}{ccccc}
 H^{d-2r}(\text{Sh}(G_{\mathbf{S}_b, \mathbf{T}_b})) & & & & \\
 \downarrow \pi_{\delta_b}^* \circ \text{Gys}_{\mathfrak{b}_{\text{res}}} & & & & \\
 H^{d-2}(\text{Sh}(G_{\mathbf{S}, \mathbf{T}})_{\delta_b})(r-1) & \xrightarrow{\text{Restr.}} & H^{d-2}(\text{Sh}(G_{\mathbf{S}, \mathbf{T}})_{\mathfrak{a}^*})(r-1) & & \\
 \downarrow \text{Gysin} & & \downarrow \text{Gysin} & & \\
 H^d(\text{Sh}(G_{\mathbf{S}, \mathbf{T}}))(r) & \xrightarrow{\text{Restr.}} & H^d(\text{Sh}(G_{\mathbf{S}, \mathbf{T}})_{\mathfrak{a}})(r) & \xrightarrow{\pi_{\mathfrak{a}, !}} & H^{d-2r}(\text{Sh}(G_{\mathbf{S}_a, \mathbf{T}_a})).
 \end{array}$$

Here, the square is commutative because the corresponding morphisms on the Shimura varieties form a Cartesian square. The diagram implies that the π -component of $\text{Res}_a \circ \text{Gys}_b$ factors through the cohomology group

$$H^{d-2}(\text{Sh}(G_{\mathbf{S}, \mathbf{T}})_{\mathfrak{a}^*})(r-1)[\pi] \cong H^{d-2(r+1)}(\text{Sh}(G_{\mathbf{S}_{a^*}, \mathbf{T}_{a^*}}))(-1)[\pi],$$

which is the π -isotypical component of the cohomology of a quaternionic Shimura variety of dimension $d - 2(r + 1)$. This means that the conclusion of Theorem 4.3(1) holds if we ever arrive in case (i) during the inductive proof.

5.9. Case (ii). This is the case when there is a basic arc δ_a in \mathbf{a} linking τ^- to τ from left to right, and hence δ_a and δ_b are the same (up to deformation of the arcs). We write δ for the periodic semi-meander for \mathbf{S} with only one arc δ_a . We write $\mathbf{a}_{\text{res}} = \mathbf{a} \setminus \delta$ for the periodic semi-meander for \mathbf{S}_δ obtained by removing δ_a from \mathbf{a} and placing its end-nodes by plus signs.

Using the discussion of Subsection 5.5, the morphism $\text{Res}_a \circ \text{Gys}_b$ is the composition from the upper-left to the upper-right going all the way around: first downwards to the bottom, then all the way to the right, and finally upwards.

$$\begin{array}{ccc}
H^{d-2r}(\text{Sh}(G_{\mathbf{S}_a, \mathbf{T}_a})) & & H^{d-2r}(\text{Sh}(G_{\mathbf{S}_b, \mathbf{T}_b})) \\
\downarrow \text{Gys}_{b_{\text{res}}} & & \uparrow \text{Res}_{a_{\text{res}}} \\
H^{d-2}(\text{Sh}(G_{\mathbf{S}_\delta, \mathbf{T}_\delta}))(r-1) & \dashrightarrow & H^{d-2}(\text{Sh}(G_{\mathbf{S}_\delta, \mathbf{T}_\delta}))(r-1) \\
\downarrow \pi_\delta^* & & \uparrow \pi_{\delta,!} \\
H^{d-2}(\text{Sh}(G_{\mathbf{S}, \mathbf{T}})_\delta)(r-1) & \xrightarrow{\text{Gysin}} & H^d(\text{Sh}(G_{\mathbf{S}, \mathbf{T}}))(r) \xrightarrow{\text{Restr.}} H^d(\text{Sh}(G_{\mathbf{S}, \mathbf{T}})_\delta)(r)
\end{array}$$

As in Subsection 5.2, the composition of the bottom line is given by the excessive intersection formula, that is to take the cup product with the first Chern class of the normal bundle of the embedding $\text{Sh}(G_{\mathbf{S}, \mathbf{T}})_\delta \hookrightarrow \text{Sh}(G_{\mathbf{S}, \mathbf{T}})$, which is $-2p^{\ell(\delta)}$ times the class of the canonical quotient bundle for the \mathbb{P}^1 -bundle given by π_δ , according to Proposition 2.31(2). Therefore, the dotted arrow in the middle is simply multiplication by $-2p^{\ell(\delta)}$. From this, we deduce that

$$(5.9.1) \quad \text{Res}_a \circ \text{Gys}_b = -2p^{\ell(\delta)} \cdot \text{Res}_{a_{\text{res}}} \circ \text{Gys}_{b_{\text{res}}},$$

where the latter morphism is constructed over the Shimura variety $\text{Sh}(G_{\mathbf{S}_\delta, \mathbf{T}_\delta})$ of lower dimension. (Here we choose the shift $\mathbf{t}'_{a'}$ for a periodic semi-meander \mathbf{a}' for $(\mathbf{S}_\delta, \mathbf{T}_\delta)$ to be $\mathbf{t}_{\delta, \tilde{\mathbf{a}}'}$, where $\tilde{\mathbf{a}}'$ is a periodic semi-meander of (\mathbf{S}, \mathbf{T}) consisting of all the arcs and semi-lines of \mathbf{a}' together with the arc δ .)

We can now complete the induction in this case, since we have already known Theorem 4.3 for $\text{Res}_{a_{\text{res}}} \circ \text{Gys}_{b_{\text{res}}}$ by induction hypothesis.

- (1) If $\langle \mathbf{a}, \mathbf{b} \rangle = 0$, then $\langle \mathbf{a}_{\text{res}}, \mathbf{b}_{\text{res}} \rangle = 0$ for simple combinatorics reasons. Then the π -isotypical component of $\text{Res}_{a_{\text{res}}} \circ \text{Gys}_{b_{\text{res}}}$ factors through the cohomology of a lower dimensional Shimura variety, so the same is true for $\text{Res}_a \circ \text{Gys}_b$.
- (2) or (3) We have $\langle \mathbf{a} | \mathbf{b} \rangle = (-2)^{m_0} v^{m_v}$ or $(-2)^{m_0} T^{m_T}$. The picture $D(\mathbf{a}_{\text{res}}, \mathbf{b}_{\text{res}})$ is given by removing from $D(\mathbf{a}, \mathbf{b})$ the contractible loop consisting of δ_a and δ_b . So we have

$$\langle \mathbf{a}_{\text{res}} | \mathbf{b}_{\text{res}} \rangle = (-2)^{-1} \langle \mathbf{a} | \mathbf{b} \rangle = \begin{cases} (-2)^{m_0-1} v^{m_v} & \text{if } r < \frac{d}{2}, \\ (-2)^{m_0-1} T^{m_T} & \text{if } r = \frac{d}{2}. \end{cases}$$

Since we have $\ell(\mathbf{a}) - \ell(\mathbf{a}_{\text{res}}) = \ell(\mathbf{b}) - \ell(\mathbf{b}_{\text{res}}) = \ell(\delta)$ and $\mathbf{t}'_{a_{\text{res}}} \mathbf{t}'_{b_{\text{res}}}^{-1} = \mathbf{t}_a \mathbf{t}_b^{-1}$, we see that $\eta_{\mathbf{S}_a, \mathbf{S}_b}$ gives the same link morphism as $\eta_{\mathbf{S}_{\delta, a_{\text{res}}}, \mathbf{S}_{\delta, b_{\text{res}}}}$ (with the same indentation and shift). By the inductive hypothesis and (5.9.1),

$$\begin{aligned}
\text{Res}_a \circ \text{Gys}_b &= -2p^{\ell(\delta)} \cdot \text{Res}_{a_{\text{res}}} \circ \text{Gys}_{b_{\text{res}}} \\
&= \begin{cases} -2p^{\ell(\delta)} \cdot (-2)^{m_0-1} \cdot p^{(\ell(\mathbf{a}_{\text{res}}) + \ell(\mathbf{b}_{\text{res}}))/2} \eta_{\mathbf{S}_{\delta, a_{\text{res}}}, \mathbf{S}_{\delta, b_{\text{res}}}}^*(z), & \text{if } r < \frac{d}{2}, \\ -2p^{\ell(\delta)} \cdot (-2)^{m_0-1} \cdot p^{(\ell(\mathbf{a}_{\text{res}}) + \ell(\mathbf{b}_{\text{res}}))/2} (T_{\mathbb{P}}/p^{g/2})^{m_T} \eta_{\mathbf{S}_{\delta, a_{\text{res}}}, \mathbf{S}_{\delta, b_{\text{res}}}}^*(z) & \text{if } r = \frac{d}{2}, \end{cases} \\
&= \begin{cases} (-2)^{m_0} \cdot p^{(\ell(\mathbf{a}) + \ell(\mathbf{b}))/2} \eta_{\mathbf{S}_a, \mathbf{S}_b}^*(z), & \text{if } r < \frac{d}{2}, \\ (-2)^{m_0} \cdot p^{(\ell(\mathbf{a}) + \ell(\mathbf{b}))/2} (T_{\mathbb{P}}/p^{g/2})^{m_T} \eta_{\mathbf{S}_a, \mathbf{S}_b}^*(z) & \text{if } r = \frac{d}{2}. \end{cases}
\end{aligned}$$

5.10. **Case (iii).** This is the case when there is an arc δ_a in \mathfrak{a} connecting τ and τ^- wrapped around the cylinder from right to left, and hence δ_a and δ_b together form a non-contractible loop in $D(\mathfrak{a}, \mathfrak{b})$. We are forced to have $d = 2r$ in this case (and hence p splits in E/F). Moreover, the arc δ_a must lie over all other arcs of \mathfrak{a} (if there is any). We now define a list of notations followed by an example.

- Let $\delta_{a,\bullet}$ (resp. $\delta_{b,\bullet}$) denote the periodic semi-meander of two nodes obtained from \mathfrak{a} (resp. \mathfrak{b}) by keeping δ_a (resp. δ_b) and its end-nodes and replacing the other nodes of \mathfrak{a} replaced by plus signs.
- Let $\mathfrak{a}_\bullet^b = \mathfrak{a} \setminus \delta_a$ denote the periodic semi-meander for \mathbf{S}_a given by removing the arc δ_a from \mathfrak{a} and replacing the nodes at τ and τ^- by plus signs.
- Let \mathfrak{a}^b denote the periodic semi-meander for \mathbf{S} given by removing the arc δ_a and adjoining two semi-lines attached to both τ and τ^- .
- Let \mathfrak{a}^* denote the semi-meander for \mathbf{S} obtained by replacing the arc δ_a in \mathfrak{a} with δ_b instead.

For example, if $\mathfrak{a} = \tau \cdots \tau^-$ and $\mathfrak{b} = \tau \cdots \tau^-$, and we choose δ_b to be the arc of \mathfrak{b} linking the first and the last nodes (τ and τ^- respectively in the pictures), then δ_a is the arc linking the first and the last nodes (but “over” all other arcs). In this case, we have

$$\begin{aligned} \delta_{a,\bullet} &= \tau \cdots \tau^-, & \delta_{b,\bullet} &= \tau \cdots \tau^-, & \mathfrak{a}_\bullet^b &= \tau \cdots \tau^-, \\ \mathfrak{a}^b &= \tau \cdots \tau^-, & \mathfrak{a}^* &= \tau \cdots \tau^-, & \text{and } \mathfrak{b}_{\text{res}} &= \tau \cdots \tau^-. \end{aligned}$$

Our goal is to prove an equality

$$(5.10.1) \quad \text{Res}_a \circ \text{Gys}_b = T_p \circ \eta_{\mathbf{S}_{a^*}, \mathbf{S}_a}^* \circ \text{Res}_{\mathfrak{a}_\bullet^b} \circ \text{Gys}_{\mathfrak{b}_{\text{res}}},$$

where $\eta_{\mathbf{S}_{a^*}, \mathbf{S}_a}^*$ is a certain link morphism associated to the trivial link $\eta_{\mathbf{S}_{a^*}, \mathbf{S}_a} : \mathbf{S}_{a^*} \rightarrow \mathbf{S}_a$ which we specify later.

Using the discussion of Subsection 5.5, we see that the morphism $\text{Res}_a \circ \text{Gys}_b$ is the composition from top-left to bottom-left of the following diagram, by going first rightward to the end, then downwards to the bottom, and finally to the left by the long arrow:

$$(5.10.2) \quad \begin{array}{ccccc} H^0(\text{Sh}(G_{\mathbf{S}_b, \mathbf{T}_b})) & & & & \\ \downarrow \text{Gys}_{\mathfrak{b}_{\text{res}}} & & & & \\ H^{d-2}(\text{Sh}(G_{\mathbf{S}_{\delta_b}, \mathbf{T}_{\delta_b}}))(\frac{d}{2}-1) & \xrightarrow{\pi_{\delta_b}^*} & H^{d-2}(\text{Sh}(G_{\mathbf{S}, \mathbf{T}})_{\delta_b})(\frac{d}{2}-1) & \xrightarrow{\text{Gysin}} & H^d(\text{Sh}(G_{\mathbf{S}, \mathbf{T}}))(\frac{d}{2}) \\ \downarrow \text{Restr.} & & \downarrow \text{Restr.} & & \downarrow \text{Restr.} \\ H^{d-2}(\text{Sh}(G_{\mathbf{S}_{\delta_b}, \mathbf{T}_{\delta_b}})_{\mathfrak{a}_\bullet^b})(\frac{d}{2}-1) & \xrightarrow{\pi_{\delta_b}^*} & H^{d-2}(\text{Sh}(G_{\mathbf{S}, \mathbf{T}})_{\mathfrak{a}^*})(\frac{d}{2}-1) & \xrightarrow{\text{Gysin}} & H^d(\text{Sh}(G_{\mathbf{S}, \mathbf{T}})_{\mathfrak{a}^b})(\frac{d}{2}) \\ \downarrow \pi_{\mathfrak{a}_\bullet^b, !} & & \downarrow \pi_{\mathfrak{a}^b, !} & & \downarrow \pi_{\mathfrak{a}^b, !} \\ H^0(\text{Sh}(G_{\mathbf{S}_{a^*}, \mathbf{T}_{a^*}})) & \xrightarrow{\pi_{\delta_b, \bullet}^*} & H^0(\text{Sh}(G_{\mathbf{S}_{a^b}, \mathbf{T}_{a^b}})_{\delta_b, \bullet}) & \xrightarrow{\text{Gysin}} & H^2(\text{Sh}(G_{\mathbf{S}_{a^b}, \mathbf{T}_{a^b}}))_{\delta_b, \bullet}(1) \\ \downarrow & & \downarrow & & \downarrow \text{Restr.} \\ H^0(\text{Sh}(G_{\mathbf{S}_a, \mathbf{T}_a})) & \xleftarrow{\pi_{\delta_a, \bullet, !}} & & & H^2(\text{Sh}(G_{\mathbf{S}_{a^b}, \mathbf{T}_{a^b}})_{\delta_a, \bullet})(1) \end{array}$$

The top two squares of the above diagram are commutative because the corresponding morphisms on the Shimura varieties form Cartesian squares. The middle rectangle of (5.10.2) is commutative by (5.6.3) (applied with our \mathfrak{a}^* , \mathfrak{a}^b , and δ_b being the \mathfrak{a} , \mathfrak{a}^b , and \mathfrak{a}^b therein, respectively). Now, for

the bottom rectangle, we are simply working with the Shimura variety $\text{Sh}(G_{\mathbf{S}_{\mathbf{a}^b}, \mathbf{T}_{\mathbf{a}^b}})$ and hence are reduced to the case of $d = 2$.

Using Subsection 5.3, we see that the dotted downward arrow on the left is exactly the operator $T_{\mathbf{p}}$ times a link morphism $\eta_{\mathbf{S}_{\mathbf{a}}, \mathbf{S}_{\mathbf{a}^*}}^*$ associated to the link $\eta_{\mathbf{S}_{\mathbf{a}}, \mathbf{S}_{\mathbf{a}^*}} : \mathbf{S}_{\mathbf{a}} \rightarrow \mathbf{S}_{\mathbf{a}^*}$, with indentation degree $-2\ell(\delta_{\mathbf{b}, \bullet})$ and shift

$$(5.10.3) \quad \varpi_{\bar{q}}^{-1} \mathbf{t}_{\mathbf{a}^b, \mathbf{a}^*}^{-1} \mathbf{t}_{\mathbf{a}^b, \mathbf{a}} = \varpi_{\bar{q}}^{-1} \mathbf{t}_{\mathbf{a}^*}^{-1} \mathbf{t}_{\mathbf{a}}.$$

To sum up, the morphism $\text{Res}_{\mathbf{a}} \circ \text{Gys}_{\mathbf{b}}$ is the same as the composition of the downward arrows on the left in (5.10.2). So we have proved (5.10.1).

We now complete the inductive proof of Theorem 4.3. The condition for case (iii) implies that we are in the setup of Theorem 4.3(3). Assume that we have $\langle \mathbf{a} | \mathbf{b} \rangle = (-2)^{m_0} T^{m_T}$. The picture $D(\mathbf{a}_{\bullet}^b, \mathbf{b}_{\text{res}})$ is given by removing from $D(\mathbf{a}, \mathbf{b})$ the non-contractible loop consisting of $\delta_{\mathbf{a}}$ and $\delta_{\mathbf{b}}$. So we have

$$\langle \mathbf{a}_{\bullet}^b | \mathbf{b}_{\text{res}} \rangle = T^{-1} \langle \mathbf{a} | \mathbf{b} \rangle = (-2)^{m_0} T^{m_T-1}.$$

By inductive hypothesis applied to the Shimura variety $\text{Sh}_{K_p}(G_{\mathbf{S}_{\delta_{\mathbf{b}}}, \mathbf{T}_{\delta_{\mathbf{b}}}})$ of lower dimension (where the shift $\mathbf{t}'_{\mathbf{a}'}$ for a periodic semi-meander \mathbf{a}' for $(\mathbf{S}_{\delta_{\mathbf{b}}}, \mathbf{T}_{\delta_{\mathbf{b}}})$ is taken to be $\mathbf{t}_{\delta_{\mathbf{b}}, \tilde{\mathbf{a}}'}$, where $\tilde{\mathbf{a}}'$ is a periodic semi-meander of (\mathbf{S}, \mathbf{T}) consisting of all the arcs and semi-lines of \mathbf{a}' together with the arc $\delta_{\mathbf{b}}$),

$$(5.10.4) \quad \text{Res}_{\mathbf{a}_{\bullet}^b} \circ \text{Gys}_{\mathbf{b}_{\text{res}}} = (-2)^{m_0} \cdot p^{(\ell(\mathbf{a}_{\bullet}^b) + \ell(\mathbf{b}_{\text{res}}))/2} (T_{\mathbf{p}}/p^{g/2})^{m_T-1} \circ \eta_{\mathbf{S}_{\mathbf{a}_{\bullet}^b}, \mathbf{S}_{\mathbf{b}_{\text{res}}}, (z')}^*,$$

where $\eta_{\mathbf{S}_{\mathbf{a}_{\bullet}^b}, \mathbf{S}_{\mathbf{b}_{\text{res}}}, (z')}^*$ is the trivial link morphism with shift

$$(5.10.5) \quad \mathbf{t}'_{\mathbf{a}_{\bullet}^b} \mathbf{t}'_{\mathbf{b}_{\text{res}}}^{-1} \varpi_{\bar{q}}^{-m_T+1} = \mathbf{t}_{\delta_{\mathbf{b}}, \mathbf{a}^*} \mathbf{t}_{\delta_{\mathbf{b}}, \mathbf{b}}^{-1} \varpi_{\bar{q}}^{-m_T+1} = \mathbf{t}_{\mathbf{a}^*} \mathbf{t}_{\mathbf{b}}^{-1} \varpi_{\bar{q}}^{-m_T+1}$$

and indentation degree $z = \ell(\mathbf{a}_{\bullet}^b) - \ell(\mathbf{b}_{\text{res}}) - (m_T - 1)g$. Combining (5.10.1) and (5.10.4) with the numerical equalities

$$\ell(\mathbf{a}_{\bullet}^b) = \ell(\mathbf{a}) - g + \ell(\delta_{\mathbf{b}, \bullet}) \quad \text{and} \quad \ell(\mathbf{b}_{\text{res}}) = \ell(\mathbf{b}) - \ell(\delta_{\mathbf{b}, \bullet}),$$

we deduce that

$$\text{Res}_{\mathbf{a}} \circ \text{Gys}_{\mathbf{b}} = (-2)^{m_0} p^{(\ell(\mathbf{a}) + \ell(\mathbf{b}))/2} (T_{\mathbf{p}}/p^{g/2})^{m_T} \circ \eta_{\mathbf{S}_{\mathbf{a}}, \mathbf{S}_{\mathbf{a}^*}}^* \circ \eta_{\mathbf{S}_{\mathbf{a}_{\bullet}^b}, \mathbf{S}_{\mathbf{b}_{\text{res}}}, (z')}^*.$$

The composition of the last two link morphism is a link morphism $\mathbf{S}_{\mathbf{a}} \rightarrow \mathbf{S}_{\mathbf{a}^*} = \mathbf{S}_{\mathbf{a}_{\bullet}^b} \rightarrow \mathbf{S}_{\mathbf{b}_{\text{res}}} = \mathbf{S}_{\mathbf{b}}$, whose indentation degree is

$$-2\ell(\delta_{\mathbf{b}, \bullet}) + z = \ell(\mathbf{a}) - \ell(\mathbf{b}) - m_T g$$

and whose shift is equal to the product of (5.10.3) and (5.10.5), or explicitly,

$$\varpi_{\bar{q}}^{-1} \mathbf{t}_{\mathbf{a}^*}^{-1} \mathbf{t}_{\mathbf{a}} \cdot \mathbf{t}_{\mathbf{a}^*} \mathbf{t}_{\mathbf{b}}^{-1} \varpi_{\bar{q}}^{-m_T+1} = \mathbf{t}_{\mathbf{a}} \mathbf{t}_{\mathbf{b}}^{-1} \varpi_{\bar{q}}^{-m_T}.$$

This completes Theorem 4.3(3) in this case.

5.11. Case (iv). Recall that $\delta_{\mathbf{b}}$ is a basic arc of \mathbf{b} linking τ with τ^- from right to left. We are looking at the situation when at least one of τ and τ^- is linked to an arc in \mathbf{a} that is not connected to the other node. We start with a long list of combinatorics construction, followed by two examples.

- Let \mathbf{a}° be the periodic semi-meander for $\mathbf{S}_{\delta_{\mathbf{b}}}$ given by first replacing the nodes τ, τ^- by plus signs, then adjoining the basic arc $\delta_{\mathbf{b}}$ to \mathbf{a} from *underneath* the band to connect to the arcs or links that are already linked to the nodes τ^-, τ , and finally continuously deform the picture so that all arcs are above the band and all semi-lines are straight. Intuitively, one can view the last step as “pulling the strings to tighten the drawing.”
- Let \mathbf{a}^* denote the periodic semi-meander for \mathbf{S} modified from \mathbf{a}^* by replacing the plus signs τ, τ^- by nodes and adjoining them by the arc $\delta_{\mathbf{b}}$.

- Let \mathfrak{a}^b denote the periodic semi-meander for \mathbf{S} that consists of two semi-lines at both τ and τ^- , all arcs in \mathfrak{a} that do not intersect with these two semi-lines, and semi-lines at the nodes that are not connected to anything above. Let $r' (< r)$ denote the number of arcs in \mathfrak{a}^b so that $\mathfrak{a}^b \in \mathfrak{B}_S^{r'}$.
- Let \mathfrak{a}_+^b denote the periodic semi-meander for \mathbf{S}_{δ_b} obtained by removing the two semi-lines at both τ and τ^- from \mathfrak{a}^b and replacing the nodes at τ, τ^- by plus signs.
- We use \mathfrak{a}_τ^b to denote the periodic semi-meander for \mathbf{S} given by replacing in \mathfrak{a}^b the two semi-lines connected to τ and τ^- by δ_b .
- We use δ_{b/a^b} to denote the periodic semi-meander for \mathbf{S}_{a^b} consisting of only one arc δ_b (and all semi-lines of \mathfrak{a}_+^b).
- We choose and fix an arc δ_a of \mathfrak{a} such that
 - Case (a). either τ is the left end-node of δ_a , or
 - Case (b). τ^- is the right end-node of δ_a .

Such an arc δ_a exists under the assumption of Case (iv) (there might be one or two such arcs).

We use τ' to denote the right endpoint of δ_a . Thus, τ' is neither τ nor τ^- in Case (a), and τ' is the same as τ^- in Case (b).

- We use $\mathfrak{a}_{\tau'}^b$ to denote the periodic semi-meander for \mathbf{S} given by deleting from \mathfrak{a}^b the two semi-lines connected to the end-nodes of δ_a and then adjoining the arc δ_a .
- We use δ_{a/a^b} to denote the periodic semi-meander for \mathbf{S}_{a^b} consisting of only one arc δ_a (and all semi-lines of $\mathfrak{a}_{\tau'}^b$).
- We use $\eta_{\mathfrak{a}_{\tau'}^b, \mathfrak{a}_\tau^b}$ to denote the link from $\mathbf{S}_{\mathfrak{a}_{\tau'}^b}$ to $\mathbf{S}_{\mathfrak{a}_\tau^b}$ given by the reduction of $D(\delta_{b/a^b}, \delta_{a/a^b})$ as defined in 3.1.
- We use $\mathfrak{a}_{\text{res}}^b$ to denote the periodic semi-meander for $\mathbf{S}_{\mathfrak{a}_{\tau'}^b}$ given by deleting all arcs in \mathfrak{a} that has already appeared in $\mathfrak{a}_{\tau'}^b$, and changing their end-nodes to plus signs.
- We use $\mathfrak{a}_{\text{res}}^\circ$ to denote the periodic semi-meander for $\mathbf{S}_{\mathfrak{a}^\circ}$ with nodes given by deleting all arcs in \mathfrak{a}° that has already appeared in \mathfrak{a}_+^b , and changing their end-nodes to plus signs.
- We use $\eta_{\mathfrak{a}, \mathfrak{a}^\star}$ to denote the link from $\mathbf{S}_\mathfrak{a}$ to $\mathbf{S}_{\mathfrak{a}^\star} = (\mathbf{S}_{\delta_b})_{\mathfrak{a}^\circ}$ which is the restriction of $\eta_{\mathfrak{a}_{\tau'}^b, \mathfrak{a}_\tau^b}$ to $\mathbf{S}_\mathfrak{a}$.
- We use $\mathfrak{b}_{\text{res}}$ to denote the semi-meander for \mathbf{S}_{δ_b} obtained by deleting the arc δ_b and replacing the nodes τ, τ^- by plus signs.

We now give two examples. In both instances, \mathfrak{b} has a basic arc connecting the node 1 with 2 (starting with node 0 on the left). So node 1 is τ^- and node 2 is τ .

Example 1: We take $\mathbf{a} = \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet$, then the arc $\delta_{\mathbf{a}}$ has to be the one connecting nodes 2 and 5 and τ' is the node 5. We are in Case (a), and we have

$$\begin{aligned}
 \delta_{\mathbf{a}} &= \begin{array}{c} \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \\ \tau^- \tau \quad \tau' \end{array}, & \mathbf{a}^b &= \begin{array}{c} \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \\ \tau^- \tau \quad \tau' \end{array}, & \mathbf{a}_+^b &= \begin{array}{c} \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \\ \tau^- \tau \quad \tau' \end{array}, \\
 \mathbf{a}_\tau^b &= \begin{array}{c} \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \\ \tau^- \tau \quad \tau' \end{array}, & \mathbf{a}_{\tau'}^b &= \begin{array}{c} \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \\ \tau^- \tau \quad \tau' \end{array}, & \delta_{\mathbf{b}/\mathbf{a}^b} &= \begin{array}{c} \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \\ \tau^- \tau \quad \tau' \end{array}, \\
 \delta_{\mathbf{a}/\mathbf{a}^b} &= \begin{array}{c} \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \\ \tau^- \tau \quad \tau' \end{array}, & \mathbf{a}_{\text{res}}^b &= \begin{array}{c} \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \\ \tau^- \tau \quad \tau' \end{array}, & \mathbf{a}^o &= \begin{array}{c} \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \\ \tau^- \tau \quad \tau' \end{array},^{22} \\
 \mathbf{a}^* &= \begin{array}{c} \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \\ \tau^- \tau \quad \tau' \end{array}, & \mathbf{a}_{\text{res}}^o &= \begin{array}{c} \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \\ \tau^- \tau \quad \tau' \end{array}, & \eta_{\mathbf{a}_{\tau'}^b, \mathbf{a}_\tau^b} &= \begin{array}{c} \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \\ \tau^- \tau \quad \tau' \end{array}, \\
 \eta_{\mathbf{a}, \mathbf{a}^*} &= \begin{array}{c} \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \\ \tau^- \tau \quad \tau' \end{array}.
 \end{aligned}$$

Example 2: We take $\mathbf{a} = \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet$, then the arc $\delta_{\mathbf{a}}$ has to be the one connecting nodes 1 and 8 through the imaginary boundary at $x = -1/2$ and $x = g - 1/2$. We are in Case (b), so $\tau' = \tau^-$ is the node 1. We have

$$\begin{aligned}
 \delta_{\mathbf{a}} &= \begin{array}{c} \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \\ \tau^- \tau \end{array}, & \mathbf{a}^b &= \begin{array}{c} \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \\ \tau^- \tau \end{array}, & \mathbf{a}_+^b &= \begin{array}{c} \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \\ \tau^- \tau \end{array}, \\
 \mathbf{a}_\tau^b &= \begin{array}{c} \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \\ \tau^- \tau \end{array}, & \mathbf{a}_{\tau'}^b &= \begin{array}{c} \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \\ \tau^- \tau \end{array}, & \delta_{\mathbf{b}/\mathbf{a}^b} &= \begin{array}{c} \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \\ \tau^- \tau \end{array}, \\
 \delta_{\mathbf{a}/\mathbf{a}^b} &= \begin{array}{c} \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \\ \tau^- \tau \end{array}, & \mathbf{a}_{\text{res}}^b &= \begin{array}{c} \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \\ \tau^- \tau \end{array}, & \mathbf{a}^o &= \begin{array}{c} \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \\ \tau^- \tau \end{array}, \\
 \mathbf{a}^* &= \begin{array}{c} \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \\ \tau^- \tau \end{array}, & \mathbf{a}_{\text{res}}^o &= \begin{array}{c} \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \\ \tau^- \tau \end{array}, & \eta_{\mathbf{a}_{\tau'}^b, \mathbf{a}_\tau^b} &= \begin{array}{c} \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \\ \tau^- \tau \end{array}, \\
 \eta_{\mathbf{a}, \mathbf{a}^*} &= \begin{array}{c} \bullet \bullet \bullet \bullet \bullet \bullet \bullet \bullet \\ \tau^- \tau \end{array}.^{23}
 \end{aligned}$$

Using the discussion in Subsection 5.5, we see that the morphism $\text{Res}_{\mathbf{a}} \circ \text{Gys}_{\mathbf{b}}$ is the composition of the following diagram from the top-left to the bottom through first $\text{Gys}_{\mathbf{b}_{\text{res}}}$ and then all the way

²²We give special shape to the arc linking nodes 5 and 8 here to remind the reader that this arc is obtained by “pulling the strings”.

²³When either τ or τ^- is connected to a semi-line, a lot of the new periodic semi-meanders constructed are either “simple” or “similar” to \mathbf{a} .

to the right and then all the way downwards, and finally through $\pi_{\delta_{a/a^b},!}$ and $\text{Res}_{a^b_{\text{res}}}$.

(5.11.1)

$$\begin{array}{ccccc}
H^{d-2r}(\text{Sh}(G_{S_b, T_b})) & & & & \\
\downarrow \text{Gys}_{b_{\text{res}}} & & & & \\
H^{d-2}(\text{Sh}(G_{S_{\delta_b}, T_{\delta_b}})) & \xrightarrow{\pi_{\delta_b}^*} & H^{d-2}(\text{Sh}(G_{S, T})_{\delta_b}) & \xrightarrow{\text{Gysin}} & H^d(\text{Sh}(G_{S, T})) \\
\downarrow \text{Restr.} & & \downarrow \text{Restr.} & & \downarrow \text{Restr.} \\
H^{d-2}(\text{Sh}(G_{S_{\delta_b}, T_{\delta_b}})_{a^b_+}) & \xrightarrow{\pi_{\delta_b}^*} & H^{d-2}(\text{Sh}(G_{S, T})_{a^b_\tau}) & \xrightarrow{\text{Gysin}} & H^d(\text{Sh}(G_{S, T})_{a^b}) \\
\downarrow \pi_{a^b_+,!} & & & & \downarrow \pi_{a^b, !} \\
H^{d-2r'-2}(\text{Sh}(G_{S_{a^b_\tau}, T_{a^b_\tau}})) & \xrightarrow{\pi_{\delta_{b/a^b}}^*} & H^{d-2r'-2}(\text{Sh}(G_{S_{a^b}, T_{a^b}})_{\delta_{b/a^b}}) & \xrightarrow{\text{Gysin}} & H^{d-2r'}(\text{Sh}(G_{S_{a^b}, T_{a^b}})) \\
\downarrow \text{Res}_{a^b_{\text{res}}} & \searrow p^{y/2} \eta_{a^b_{\tau'}, a^b_\tau}^* & & & \downarrow \text{Restr.} \\
H^{d-2r}(\text{Sh}(G_{S_{a^*}, T_{a^*}})) & & H^{d-2r'-2}(\text{Sh}(G_{S_{a^b_{\tau'}}, T_{a^b_{\tau'}}})) & \xleftarrow{\pi_{\delta_{a/a^b},!}} & H^{d-2r'}(\text{Sh}(G_{S_{a^b}, T_{a^b}})_{\delta_{a/a^b}}) \\
& \searrow p^{(x+y)/2} \eta_{a, a^*}^* & \downarrow \text{Res}_{a^b_{\text{res}}} & & \\
& & H^{d-2r}(\text{Sh}(G_{S_a, T_a})) & &
\end{array}$$

Here, the numbers x, y and the link morphisms $\eta_{a^b_{\tau'}, a^b_\tau}^*$ and η_{a, a^*}^* will be defined explicitly later. For simplicity, we have omitted the Tate twists from the notation, and each cohomology group $H^a(\star)$ should be understood as $H^a(\star)(b)$ with $a - 2b = d - 2r$; for instance, $H^{d-2r'-2}(\text{Sh}(G_{S_{a^b_{\tau'}}, T_{a^b_{\tau'}}}))$ should be understood as $H^{d-2r'-2}(\text{Sh}(G_{S_{a^b_{\tau'}}, T_{a^b_{\tau'}}}))(r - r' - 1)$.

We now explain the commutativity of this diagram. The top two squares are commutative because the corresponding morphisms on the Shimura varieties form Cartesian squares. The commutativity of the middle rectangle follows from that of (5.6.3) (applied with our a^b_τ and a^b being the a and a^b therein, respectively). The commutativity of the lower trapezoid will follow from applying Subsection 5.4 (applied to the Shimura variety $\text{Sh}(G_{S_{a^b}, T_{a^b}})$ with our δ_{a/a^b} and δ_{b/a^b} being the a and b therein) once we have clarified the meaning of y and $\eta_{a^b_{\tau'}, a^b_\tau}^*$ in 5.12. Finally, the commutativity of the bottom parallelogram will be justified in Subsection 5.13 and Lemma 5.14 later.

To sum up, the morphism $\text{Res}_a \circ \text{Gys}_b$ will be the composition of (5.11.1) from the top-left to the bottom by first going all the way down and through η_{a, a^*}^* . This gives the following equality

$$(5.11.2) \quad \text{Res}_a \circ \text{Gys}_b = p^{(x+y)/2} \eta_{a, a^*}^* \circ \text{Res}_{a^*} \circ \text{Gys}_{b_{\text{res}}},$$

using which we will complete the inductive proof of Theorem 4.3 in Case (iv), as we shall explain in 5.15.

5.12. Link morphism $\eta_{a^b_{\tau'}, a^b_\tau}$. Now, let us get to the details, starting with the link morphism associated with $\eta_{a^b_{\tau'}, a^b_\tau}$. We distinguish the two cases:

Case (a). Suppose that the left node of δ_a is τ . Example 1 above falls into this case. All the curves in the link $\eta_{a^b_{\tau'}, a^b_\tau}$ are semi-lines, except for one which turns to the right and we denote by ξ . The curve ξ sends τ^- to τ' . By Theorem 2.32(2), there exists a link morphism $\eta_{a^b_{\tau'}, a^b_\tau, \#}$ with indentation degree $\ell(\delta_a) - \ell(\delta_b)$ and shift $t_{a^b_{\tau'}} t_{a^b_\tau}^{-1}$ (and also a link morphism on the

local system as in Theorem 2.32(2)(d)), which fits into the following commutative diagram

$$\begin{array}{ccccc}
 \mathrm{Sh}(G_{\mathbf{S}_{a^b}, \mathbf{T}_{a^b}})_{\delta_a} & \xleftarrow{\quad} & \mathrm{Sh}(G_{\mathbf{S}_{a^b}, \mathbf{T}_{a^b}})_{\{\tau', \tau\}} & \xrightarrow{\quad} & \mathrm{Sh}(G_{\mathbf{S}_{a^b}, \mathbf{T}_{a^b}})_{\delta_b} \\
 \pi_{\delta_a} \downarrow & \nearrow \cong & & & \downarrow \pi_{\delta_b} \\
 \mathrm{Sh}(G_{\mathbf{S}_{a_{\tau'}^b}, \mathbf{T}_{a_{\tau'}^b}}) & \xrightarrow{\eta_{a_{\tau'}^b, a_{\tau'}^b, \#}} & & & \mathrm{Sh}(G_{\mathbf{S}_{a_{\tau}^b}, \mathbf{T}_{a_{\tau}^b}})
 \end{array}$$

By Theorem 2.32(2)(c) $\eta_{a_{\tau'}^b, a_{\tau'}^b, \#}$ is finite flat of degree p^y with $y := v(\eta_{a_{\tau'}^b, a_{\tau'}^b, \#}) = \ell(\delta_a) + \ell(\delta_b)$.

Let $\eta_{a_{\tau'}^b, a_{\tau}^b}^* : H^{d-2r'-2}(\mathrm{Sh}(G_{\mathbf{S}_{a_{\tau'}^b}, \mathbf{T}_{a_{\tau'}^b}})) \rightarrow H^{d-2r'-2}(\mathrm{Sh}(G_{\mathbf{S}_{a_{\tau}^b}, \mathbf{T}_{a_{\tau}^b}}))$ denote the induced link homomorphism on the cohomology groups. By the same argument as in 5.4, we see that the trapezoid in the diagram (5.11.1) is commutative.

Case (b). Suppose now that the right node of δ_a is τ^- . Example 2 above falls into this case. Then the only genuine curve in the link $\eta_{a_{\tau'}^b, a_{\tau}^b}$ is turning to the left with displacement $y = \ell(\delta_a) + \ell(\delta_b)$. Let $\eta_{a_{\tau}^b, a_{\tau'}^b}$ be the inverse link of $\eta_{a_{\tau'}^b, a_{\tau}^b}$. Applying the discussion in *Case (a)* to $\eta_{a_{\tau}^b, a_{\tau'}^b}$, one gets a link morphism $\eta_{a_{\tau}^b, a_{\tau'}^b, \#} : \mathrm{Sh}(G_{\mathbf{S}_{a_{\tau}^b}, \mathbf{T}_{a_{\tau}^b}}) \rightarrow \mathrm{Sh}(G_{\mathbf{S}_{a_{\tau'}^b}, \mathbf{T}_{a_{\tau'}^b}})$ of indentation degree $\ell(\delta_b) - \ell(\delta_a)$ and shift $\mathbf{t}_{a_{\tau}^b} \mathbf{t}_{a_{\tau'}^b}^{-1}$. By Lemma 2.29, we get a well-defined link morphism on the cohomology groups

$$(5.12.1) \quad \eta_{a_{\tau'}^b, a_{\tau}^b}^* = (\eta_{a_{\tau}^b, a_{\tau'}^b}^*)^{-1} = p^{-y/2} \mathrm{Tr}_{\eta_{a_{\tau'}^b, a_{\tau}^b, \#}} : H_{\mathrm{et}}^{d-2r'-2}(\mathrm{Sh}(G_{\mathbf{S}_{a_{\tau'}^b}, \mathbf{T}_{a_{\tau'}^b}})) \rightarrow H_{\mathrm{et}}^{d-2r'-2}(\mathrm{Sh}(G_{\mathbf{S}_{a_{\tau}^b}, \mathbf{T}_{a_{\tau}^b}}))$$

of indentation degree $\ell(a) - \ell(b)$ associated to the link $\eta_{a_{\tau'}^b, a_{\tau}^b}$ and shift $\mathbf{t}_{a_{\tau'}^b} \mathbf{t}_{a_{\tau}^b}^{-1}$. Now the argument as in 5.4 proves the commutativity of the trapezoid in the diagram (5.11.1).

5.13. Commutativity of the parallelogram in (5.11.1). We continue the discussion above by separating the two cases.

Case (a). Consider the $(r - r' - 1)$ -th iterated \mathbb{P}^1 -bundle $\pi_{a_{\mathrm{res}}^b} : \mathrm{Sh}(G_{\mathbf{S}_{a_{\tau'}^b}, \mathbf{T}_{a_{\tau'}^b}})_{a_{\mathrm{res}}^b} \rightarrow \mathrm{Sh}(G_{\mathbf{S}_a, \mathbf{T}_a})$. By applying repeatedly [TX13⁺a, Proposition 7.17] and Construction 2.15, one produces a commutative diagram:

$$\begin{array}{ccccccc}
 \mathrm{Sh}(G_{\mathbf{S}_{a_{\tau'}^b}, \mathbf{T}_{a_{\tau'}^b}})_{a_{\mathrm{res}}^b} & \xrightarrow{\pi_1^b} & X_1 & \xrightarrow{\pi_2^b} & X_2 & \longrightarrow \cdots \longrightarrow & X_{r-r'-2} \xrightarrow{\pi_{r-r'-1}^b} \mathrm{Sh}(G_{\mathbf{S}_a, \mathbf{T}_a}) \\
 \downarrow \eta_{a_{\tau'}^b, a_{\tau}^b, \#} & & \downarrow \eta_{1, \#} & & \downarrow \eta_{2, \#} & & \downarrow \eta_{r-r'-2, \#} \\
 \mathrm{Sh}(G_{\mathbf{S}_{a_{\tau'}^b}, \mathbf{T}_{a_{\tau'}^b}})_{a_{\mathrm{res}}^{\circ}} & \xrightarrow{\pi_1^{\circ}} & Y_1 & \xrightarrow{\pi_2^{\circ}} & Y_2 & \longrightarrow \cdots \longrightarrow & Y_{r-r'-2} \xrightarrow{\pi_{r-r'-1}^{\circ}} \mathrm{Sh}(G_{\mathbf{S}_{a^*}, \mathbf{T}_{a^*}}),
 \end{array}$$

where π_i^b and π_i° are all \mathbb{P}^1 -fibrations, the vertical arrows are link morphisms (associated to certain links), and the composition of top (resp. bottom) horizontal arrows is $\pi_{a_{\mathrm{res}}^b}$ (resp. $\pi_{a_{\mathrm{res}}^{\circ}}$). There exist at the same time link morphisms $\eta_i^{\#}$ and $\eta_{a, a^{\circ}}^{\#}$ on the étale local systems satisfying a similar commutative diagram. We explain now how to construct $\eta_{1, \#} : X_1 \rightarrow Y_1$, one chooses a basic arc δ_c in a_{res}^b . Let $a_{\mathrm{res}, 1}^b$ be the periodic semi-meander obtained by removing δ_c from a_{res}^b and replacing the end-nodes of δ_c by plus signs, and $a_{\tau', 1}^b$ be the periodic semi-meander obtained by removing from $a_{\tau'}^b$ the semi-lines at the end-nodes of δ_c and adjoining δ_c . Put $X_1 := \mathrm{Sh}(G_{\mathbf{S}_{a_{\tau', 1}^b}, \mathbf{T}_{a_{\tau', 1}^b}})_{a_{\mathrm{res}, 1}^b}$, and denote by

$$\pi_1^b : \mathrm{Sh}(G_{\mathbf{S}_{a_{\tau', 1}^b}, \mathbf{T}_{a_{\tau', 1}^b}})_{a_{\mathrm{res}}^b} \longrightarrow X_1$$

the \mathbb{P}^1 -fibration given by the arc δ_c . Let δ_{c° denote the arc $\eta_{a_{\tau'}, a_\tau^b}(\delta_c)$ obtained by extending δ_c using the curves of $\eta_{a_{\tau'}, a_\tau^b}$ at the end-nodes of δ_c . This δ_{c° is a basic arc in $\mathbf{a}_{\text{res}}^\circ$. We define periodic semi-meanders $\mathbf{a}_{\tau,1}^b$ and $\mathbf{a}_{\text{res},1}^\circ$ in the same way as $\mathbf{a}_{\tau',1}^b$ and $\mathbf{a}_{\text{res},1}^b$ with δ_c replaced by δ_{c° . Then we have a \mathbb{P}^1 -fibration

$$\pi_1^\circ : \text{Sh}(G_{\mathbf{S}_{a_{\tau'}^b, \mathbf{T}_{a_\tau^b}}})_{\mathbf{a}_{\text{res}}^\circ} \longrightarrow Y_1 := \text{Sh}(G_{\mathbf{S}_{a_{\tau,1}^b, \mathbf{T}_{a_{\text{res},1}^\circ}}})_{\mathbf{a}_{\text{res},1}^\circ}.$$

If $\eta_1 : \mathbf{S}_{a_{\tau',1}^b} \rightarrow \mathbf{S}_{a_{\tau,1}^b}$ denotes the link induced by $\eta_{a_{\tau'}, a_\tau^b}$, then [TX13⁺a, Proposition 7.17] and Construction 2.15 implies the existence of the link morphism $\eta_{1,\#}$ which fits into the left commutative square of (5.13.1). This finishes the construction of X_1 and Y_1 . The induced link η_1 has the same property as $\eta_{a_{\tau'}, a_\tau^b}$, namely all the curves of η_1 are semi-lines except possibly for one turning to the right. The rest of (5.13.1) can be constructed inductively in a similar way.

Since we require the diagram (5.13.1) to be commutative, by Remark 2.16, the link morphism $\eta_{a,a^*,\#}$ has shift

$$\mathbf{t}_{a_{\tau'}, a} \cdot \mathbf{t}_{a_\tau^b, a^*}^{-1} \cdot (\text{shift of } \eta_{a_{\tau'}, a_\tau^b}) = \mathbf{t}_a \mathbf{t}_{a^*}^{-1}.$$

Moreover, the indentation degree of $\eta_{a,a^*,\#}$ is $\ell(\delta_a) - \ell(\delta_b) + \ell(\mathbf{a}_{\text{res}}^b) - \ell(\mathbf{a}_{\text{res}}^\circ)$ if \mathfrak{p} splits in E/F and degree 0 if \mathfrak{p} is inert in E/F . Note also that even through each $\eta_{i,\#}$ is not unique (since there are many ways to choose a basic arc of $\mathbf{a}_{\text{res}}^b$ for instance), the final link morphism $\eta_{a,a^*,\#}$ is uniquely determined by the uniqueness of link morphisms. By [TX13⁺a, Proposition 7.17(3)] and Construction 2.15, $\eta_{a,a^*,\#}$ is finite flat of degree $p^{v(\eta_{a,a^*})}$. We have thus the normalized link morphisms $\eta_{a_{\tau'}, a_\tau^b}^*$ and η_{a,a^*}^* on the corresponding cohomology groups as defined in (2.25.1) induced by $(\eta_{a_{\tau'}, a_\tau^b}, \eta_{a_{\tau'}, a_\tau^b}^\#)$ and $(\eta_{a,a^*}^\#, \eta_{a,a^*}^\#)$ respectively.

Case (b) Suppose now that the right node of δ_a is τ^- . Applying the discussion in *Case (a)* to the inverse link $\eta_{a_{\tau'}, a_\tau^b}$, one gets a link morphism $\eta_{a^*, a,\#} : \text{Sh}(G_{\mathbf{S}_{a^*, \mathbf{T}_{a^*}}}) \rightarrow \text{Sh}(G_{\mathbf{S}_a, \mathbf{T}_a})$ associated to the inverse link of η_{a,a^*} of indentation degree $\ell(\delta_b) - \ell(\delta_a) + \ell(\mathbf{a}_{\text{res}}^\circ) - \ell(\mathbf{a}_{\text{res}}^b)$, and shift $\mathbf{t}_a^{-1} \mathbf{t}_{a^*}$. By Lemma 2.29, we get a well-defined link morphism on the cohomology groups

$$(5.13.2) \quad \eta_{a,a^*}^* = (\eta_{a^*, a,\#})^{-1} = p^{v(\eta_{a,a^*})/2} \text{Tr}_{\eta_{a^*, a,\#}} : H^{d-2r}(\text{Sh}_{G_{\mathbf{S}, \mathbf{T}}}) \rightarrow H^{d-2r}(\text{Sh}_{G_{\mathbf{S}_{a^*}, \mathbf{T}_{a^*}}})$$

of indentation degree $\ell(\delta_a) - \ell(\delta_b) + \ell(\mathbf{a}_{\text{res}}^b) - \ell(\mathbf{a}_{\text{res}}^\circ)$ and shift $\mathbf{t}_a \mathbf{t}_{a^*}^{-1}$ associated to the link η_{a,a^*} .

Lemma 5.14. *Under the above notation, put $x = \ell(\mathbf{a}_{\text{res}}^b) - \ell(\mathbf{a}_{\text{res}}^\circ)$ and $y = \ell(\delta_a) + \ell(\delta_b)$. Then in both Case (a) and Case (b) above, one has a commutative diagram of cohomology groups:*

$$\begin{array}{ccccc} H_{\text{et}}^{d-2r'-2}(\text{Sh}(G_{\mathbf{S}_{a_{\tau'}^b, \mathbf{T}_{a_\tau^b}}})) & \xrightarrow{\text{Rest.}} & H_{\text{et}}^{d-2r'-2}(\text{Sh}(G_{\mathbf{S}_{a_{\tau'}^b, \mathbf{T}_{a_\tau^b}}})_{\mathbf{a}_{\text{res}}^*}) & \xrightarrow{\pi_{\mathbf{a}_{\text{res}}^\circ, !}} & H_{\text{et}}^{d-2r}(\text{Sh}(G_{\mathbf{S}_{a^*, \mathbf{T}_{a^*}}})) \\ \downarrow p^{y/2} \eta_{a_{\tau'}, a_\tau^b}^* & & \downarrow p^{y/2} \eta_{a_{\tau'}, a_\tau^b}^* & & \downarrow p^{(x+y)/2} \eta_{a,a^*}^* \\ H_{\text{et}}^{d-2r'-2}(\text{Sh}(G_{\mathbf{S}_{a_{\tau'}^b, \mathbf{T}_{a_\tau^b}}})) & \xrightarrow{\text{Rest.}} & H_{\text{et}}^{d-2r'-2}(\text{Sh}(G_{\mathbf{S}_{a_{\tau'}^b, \mathbf{T}_{a_\tau^b}}})_{\mathbf{a}_{\text{res}}^b}) & \xrightarrow{\pi_{\mathbf{a}_{\text{res}}^b, !}} & H_{\text{et}}^{d-2r}(\text{Sh}(G_{\mathbf{S}_a, \mathbf{T}_a})). \end{array}$$

Note that the composite of the top (resp. bottom) two horizontal morphisms above is exactly $\text{Res}_{\mathbf{a}_{\text{res}}^\circ}$ (resp. $\text{Res}_{\mathbf{a}_{\text{res}}^b}$). So this verifies the commutativity of the parallelogram in (5.11.1).

Proof. The commutativity of the left square is evident. We check the commutativity of the right hand side square case by case.

Suppose first we are in *Case (a)*, i.e. the left end-node of δ_a is τ . We distinguish three subcases:

Case (a1) τ^- is linked to a semi-line in \mathbf{a} . Then both $\mathbf{a}_{\text{res}}^b$ and $\mathbf{a}_{\text{res}}^\circ$ contain no arcs. It follows that $x = 0$, and $\pi_{\mathbf{a}_{\text{res}}^b}$ and $\pi_{\mathbf{a}_{\text{res}}^\circ}$ are isomorphisms. In this case, the commutativity of the right hand side square is trivial.

Case (a2) τ^- is the left end-node of an arc in \mathbf{a} . Example 1 above falls into this case. It is easy to see that $x = y$, and that the link $\eta_{\mathbf{a}, \mathbf{a}^\star}$ contains only semi-lines. By [TX13⁺a, Proposition 7.17(3)] and Construction 2.15, $\eta_{\mathbf{a}, \mathbf{a}^\star, \#}$ is an isomorphism. Consider the commutative diagram (5.13.1). Both top and bottom rows are factorizations of $(r - r' - 1)$ -th iterated \mathbb{P}^1 -bundles as in (4.1.2). For each $1 \leq i \leq r - r' - 1$, let $\xi'_i \in H_{\text{et}}^2(\text{Sh}_{K_p}(G_{\mathbf{S}_{\mathbf{a}_{\tau'}^b, \mathbf{T}_{\mathbf{a}_{\tau'}^b}})_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(1))$ (resp. $\xi_i \in H_{\text{et}}^2(\text{Sh}_{K_p}(G_{\mathbf{S}_{\mathbf{a}_\tau^b, \mathbf{T}_{\mathbf{a}_\tau^b}})_{\overline{\mathbb{F}}_p}, \overline{\mathbb{Q}}_\ell(1))$) be the inverse image of the first Chern class of the tautological quotient line bundle of π_i^b (resp. π_i°) as considered in Subsection 4.1. Note that the only curve in $\eta_{\mathbf{a}_{\tau'}, \mathbf{a}_\tau^b}$ links the *left* end-node of an arc of $\mathbf{a}_{\text{res}}^b$ to the *left* end-node of an arc of $\mathbf{a}_{\text{res}}^\circ$. Then by applying iteratively [TX13⁺a, Proposition 7.17(3)] and Construction 2.15, there exists a unique integer i_0 with $1 \leq i_0 \leq r - r' - 1$ such that $\eta_{\mathbf{a}_{\tau'}, \mathbf{a}_\tau^b, \#}^*(\xi_{i_0}) = p^y \xi'_{i_0}$, and $\eta_{\mathbf{a}_{\tau'}, \mathbf{a}_\tau^b, \#}^*(\xi_i) = \xi'_i$ for all $i \neq i_0$. Let

$$z = \sum_{1 \leq j \leq r - r' - 1} \left(\sum_{1 \leq i_1 < \dots < i_j \leq r - r' - 1} \pi_{\mathbf{a}_{\text{res}}^\circ}^*(z_{i_1, \dots, i_j}) \cup \xi'_{i_1} \cup \dots \cup \xi'_{i_j} \right)$$

be an element of $H^{d-2r'-2}(\text{Sh}(G_{\mathbf{S}_{\mathbf{a}_\tau^b, \mathbf{T}_{\mathbf{a}_\tau^b}})_{\mathbf{a}_{\text{res}}^\circ})$ with $z_{i_1, \dots, i_j} \in H^{d-2r'-2-2j}(\text{Sh}(G_{\mathbf{S}_{\mathbf{a}^\star, \mathbf{T}_{\mathbf{a}^\star}}))$. Then one has

$$\begin{aligned} \pi_{\mathbf{a}_{\text{res}}^b, !}^*(p^{y/2} \eta_{\mathbf{a}_{\tau'}, \mathbf{a}_\tau^b}^*(z)) &= \pi_{\mathbf{a}_{\text{res}}^b, !}^* \eta_{\mathbf{a}_{\tau'}, \mathbf{a}_\tau^b, \#}^*(z) \\ &= p^y \pi_{\mathbf{a}_{\text{res}}^b, !}^* \left(\sum (\eta_{\mathbf{a}_{\tau'}, \mathbf{a}_\tau^b, \#}^*(\pi_{\mathbf{a}_{\text{res}}^\circ}^*(z_{i_1, \dots, i_j})) \cup \xi'_{i_1} \cup \dots \cup \xi'_{i_j}) \right) \\ &= p^y \pi_{\mathbf{a}_{\text{res}}^b, !}^* \left(\sum (\pi_{\mathbf{a}_{\text{res}}^b}^*(\eta_{\mathbf{a}, \mathbf{a}^\star, \#}^*(z_{i_1, \dots, i_j})) \cup \xi'_{i_1} \cup \dots \cup \xi'_{i_j}) \right) \\ &= p^y \eta_{\mathbf{a}, \mathbf{a}^\star, \#}^*(z_{1, \dots, r-r'-1}) = p^{(x+y)/2} \eta_{\mathbf{a}, \mathbf{a}^\star}^*(\pi_{\mathbf{a}_{\text{res}}^\circ, !}^*(z)), \end{aligned}$$

where the forth and fifth equalities use the formula (4.1.3). This shows the commutativity of the right square in the Lemma.

Case (a3) τ^- is the right end-node of an arc in \mathbf{a} . Then $x = -y$ and $\eta_{\mathbf{a}, \mathbf{a}^\star}$ contains only semi-lines. Hence, $\eta_{\mathbf{a}, \mathbf{a}^\star, \#}$ is an isomorphism as in *Case (a2)*. We want to show

$$\eta_{\mathbf{a}, \mathbf{a}^\star}^* \circ \pi_{\mathbf{a}_{\text{res}}^\circ, !}^* = \pi_{\mathbf{a}_{\text{res}}^b, !}^* \circ (p^{y/2} \eta_{\mathbf{a}_{\tau'}, \mathbf{a}_\tau^b}^*).$$

The argument is quite similar to that of *Case (a2)*. Let ξ_i, ξ'_i be as defined in *Case (a2)* for $1 \leq i \leq r - r' - 1$. Then by [TX13⁺a, Proposition 7.17(3)], we have $\eta_{\mathbf{a}_{\tau'}, \mathbf{a}_\tau^b, \#}^*(\xi_i) = \xi'_i$ for all $1 \leq i \leq r - r' - 1$ (this differs from the situation of *Case (a2)* because the unique curve in $\eta_{\mathbf{a}_{\tau'}, \mathbf{a}_\tau^b}$ links the *right* end-node of an arc of $\mathbf{a}_{\text{res}}^b$ to the *right* end-node of an arc of $\mathbf{a}_{\text{res}}^\circ$). Then the rest of the computation is the same as in *Case (a2)*.

Consider now *Case (b)*, i.e. the right end-node of $\delta_{\mathbf{a}}$ is τ^- . Symmetrically, we have three subcases:

Case (b1) τ is linked to a semi-line in \mathbf{a} . Then as in *Case (a1)*, we have $x = 0$, and $\pi_{\mathbf{a}_{\text{res}}^b}$ and $\pi_{\mathbf{a}_{\text{res}}^\circ}$ are both isomorphisms. The commutativity of the right hand side square is trivial.

Case (b2) τ is the left end-node of an arc in \mathbf{a} . Then $x = -y$, and $\eta_{\mathbf{a}, \mathbf{a}^\star}$ contains only semi-lines. Hence, $\eta_{\mathbf{a}, \mathbf{a}^\star, \#}$ is an isomorphism as in *Case (a2)*. By (5.12.1) and (5.13.2), the desired commutativity is equivalent to

$$\text{Tr}_{\eta_{\mathbf{a}^\star, \mathbf{a}, \#}} \circ \pi_{\mathbf{a}_{\text{res}}^\circ, !}^* = \pi_{\mathbf{a}_{\text{res}}^b, !}^* \circ \text{Tr}_{\eta_{\mathbf{a}_{\tau'}, \mathbf{a}_\tau^b, \#}},$$

which is an easy consequence of the compatibility of trace maps with composition.

Case (b3) τ is the right end-node of an arc in \mathbf{a} . Then $x = y$, and $\eta_{\mathbf{a}^*, \mathbf{a}, \#}$ is an isomorphism as in *Case (a2)*. The desired commutativity is equivalent to

$$\pi_{\mathbf{a}_{\text{res}}^b, !} \circ p^{y/2} (\eta_{\mathbf{a}_{\tau'}^b, \mathbf{a}_{\tau'}^b})^{-1} = p^y (\eta_{\mathbf{a}^*, \mathbf{a}})^{-1} \circ \pi_{\mathbf{a}_{\text{res}}^{\circ}, !} \iff \eta_{\mathbf{a}^*, \mathbf{a}} \circ \pi_{\mathbf{a}_{\text{res}}^b, !} = \pi_{\mathbf{a}_{\text{res}}^{\circ}, !} \circ (p^{y/2} \eta_{\mathbf{a}_{\tau'}^b, \mathbf{a}_{\tau'}^b}).$$

Thus the situation is exactly the same as *Case (a3)* above (except for switching the roles of $\text{Sh}(G_{\mathbf{S}_{\mathbf{a}_{\tau'}^b, \mathbf{T}_{\mathbf{a}_{\tau'}^b}})$ and $\text{Sh}(G_{\mathbf{S}_{\mathbf{a}_{\tau'}^b, \mathbf{T}_{\mathbf{a}_{\tau'}^b}})$), and we conclude by the same arguments. \square

5.15. Finish of the proof in Case (iv). We are now in position to complete the inductive proof of Theorem 2.32 in Case (iv). We have shown the commutativity of the diagram (5.11.1), from which we deduce (5.11.2):

$$\text{Res}_{\mathbf{a}} \circ \text{Gys}_{\mathbf{b}} = p^{(x+y)/2} \eta_{\mathbf{a}, \mathbf{a}^*}^* \circ \text{Res}_{\mathbf{a}^{\circ}} \circ \text{Gys}_{\mathbf{b}_{\text{res}}},$$

where $\eta_{\mathbf{a}, \mathbf{a}^*}^*$ is the link homomorphism associated to the link $\eta_{\mathbf{a}, \mathbf{a}^*} : \mathbf{S}_{\mathbf{a}} \rightarrow \mathbf{S}_{\mathbf{a}^*}$ with

- indentation $\ell(\delta_{\mathbf{a}}) - \ell(\delta_{\mathbf{b}}) + \ell(\mathbf{a}_{\text{res}}^b) - \ell(\mathbf{a}_{\text{res}}^{\circ})$ if p splits in E/F and trivial if p is inert in E/F ,
- and shift $\mathbf{t}_{\mathbf{a}} \mathbf{t}_{\mathbf{a}^*}^{-1}$.

Before proceeding, we point out the following equality of shifts which we shall use later:

$$(5.15.1) \quad \mathbf{t}_{\mathbf{a}} \mathbf{t}_{\mathbf{a}^*}^{-1} \cdot \mathbf{t}_{\delta_{\mathbf{b}}, \mathbf{a}^*} \mathbf{t}_{\delta_{\mathbf{b}}, \mathbf{b}}^{-1} = \mathbf{t}_{\mathbf{a}} \mathbf{t}_{\mathbf{b}}^{-1}.$$

Also, we point out that our decomposition of periodic semi-meanders gives a numerical equalities of spans:

$$(5.15.2) \quad \ell(\mathbf{a}) = \ell(\mathbf{a}_+^b) + \ell(\delta_{\mathbf{a}/\mathbf{a}^b}) + \ell(\mathbf{a}_{\text{res}}^b), \quad \ell(\mathbf{a}^{\circ}) = \ell(\mathbf{a}_+^b) + \ell(\mathbf{a}_{\text{res}}^{\circ}), \quad \text{and} \quad \ell(\mathbf{b}) = \ell(\delta_{\mathbf{b}}) + \ell(\mathbf{b}_{\text{res}}).$$

This (and the trivial equality $\ell(\delta_{\mathbf{a}}) = \ell(\delta_{\mathbf{a}/\mathbf{a}^b})$) implies that the indentation degree of $\eta_{\mathbf{a}, \mathbf{a}^*}^*$ when p splits in E/F , is equal to

$$(5.15.3) \quad \ell(\delta_{\mathbf{a}}) - \ell(\delta_{\mathbf{b}}) + (\ell(\mathbf{a}_{\text{res}}^b) - \ell(\mathbf{a}_{\text{res}}^{\circ})) = \ell(\mathbf{a}) - \ell(\mathbf{b}) - (\ell(\mathbf{a}^{\circ}) - \ell(\mathbf{b}_{\text{res}})).$$

Similarly, (5.15.2) also implies that

$$(5.15.4) \quad x + y = \ell(\mathbf{a}_{\text{res}}^b) - \ell(\mathbf{a}_{\text{res}}^{\circ}) + \ell(\delta_{\mathbf{a}}) + \ell(\delta_{\mathbf{b}}) = \ell(\mathbf{a}) + \ell(\mathbf{b}) - (\ell(\mathbf{a}^{\circ}) + \ell(\mathbf{b}_{\text{res}})).$$

Now we separate the discussion according to $\langle \mathbf{a} | \mathbf{b} \rangle$.

- (1) If $\langle \mathbf{a}, \mathbf{b} \rangle = 0$, then $\langle \mathbf{a}^{\circ} | \mathbf{b}_{\text{res}} \rangle = 0$ for simple combinatorics reasons. Then the π -isotypical component of $\text{Res}_{\mathbf{a}^{\circ}} \circ \text{Gys}_{\mathbf{b}_{\text{res}}}$ factors through the cohomology of a lower dimensional Shimura variety, so the same is true for $\text{Res}_{\mathbf{a}} \circ \text{Gys}_{\mathbf{b}}$.
- (2) or (3) We have $\langle \mathbf{a} | \mathbf{b} \rangle = (-2)^{m_0} v^{m_v}$ or $(-2)^{m_0} T^{m_T}$. The picture $D(\mathbf{a}^{\circ}, \mathbf{b}_{\text{res}})$ can be identified with the picture $D(\mathbf{a}, \mathbf{b})$ after deforming some curves (“pulling strings”). In particular, we have $\langle \mathbf{a}^{\circ} | \mathbf{b}_{\text{res}} \rangle = \langle \mathbf{a} | \mathbf{b} \rangle$. By the inductive hypothesis for the pair $(\mathbf{S}_{\delta_{\mathbf{b}}}, \mathbf{T}_{\delta_{\mathbf{b}}})$ ²⁴ and (5.11.2), we have

$$\begin{aligned} \text{Res}_{\mathbf{a}} \circ \text{Gys}_{\mathbf{b}} &= p^{(x+y)/2} \eta_{\mathbf{a}, \mathbf{a}^*}^* \circ \text{Res}_{\mathbf{a}^{\circ}} \circ \text{Gys}_{\mathbf{b}_{\text{res}}} \\ &= \begin{cases} p^{(x+y)/2} \eta_{\mathbf{a}, \mathbf{a}^*}^* \circ (-2)^{m_0} \cdot p^{(\ell(\mathbf{a}^{\circ}) + \ell(\mathbf{b}_{\text{res}}))/2} \eta_{\mathbf{S}_{\delta_{\mathbf{b}}}, \mathbf{a}^{\circ}, \mathbf{S}_{\delta_{\mathbf{b}}}, \mathbf{b}_{\text{res}}}^*, & \text{if } r < \frac{d}{2}, \\ p^{(x+y)/2} \eta_{\mathbf{a}, \mathbf{a}^*}^* \circ (-2)^{m_0} \cdot p^{(\ell(\mathbf{a}^{\circ}) + \ell(\mathbf{b}_{\text{res}}))/2} (T_{\mathbf{p}}/p^{g/2})^{m_T} \eta_{\mathbf{S}_{\delta_{\mathbf{b}}}, \mathbf{a}^{\circ}, \mathbf{S}_{\delta_{\mathbf{b}}}, \mathbf{b}_{\text{res}}}^*, & \text{if } r = \frac{d}{2}, \end{cases} \\ (5.15.4) \quad &= \begin{cases} (-2)^{m_0} \cdot p^{(\ell(\mathbf{a}) + \ell(\mathbf{b}))/2} \eta_{\mathbf{a}, \mathbf{a}^*}^* \circ \eta_{\mathbf{S}_{\delta_{\mathbf{b}}}, \mathbf{a}^{\circ}, \mathbf{S}_{\delta_{\mathbf{b}}}, \mathbf{b}_{\text{res}}}^*, & \text{if } r < \frac{d}{2}, \\ (-2)^{m_0} \cdot p^{(\ell(\mathbf{a}) + \ell(\mathbf{b}))/2} (T_{\mathbf{p}}/p^{g/2})^{m_T} \eta_{\mathbf{a}, \mathbf{a}^*}^* \circ \eta_{\mathbf{S}_{\delta_{\mathbf{b}}}, \mathbf{a}^{\circ}, \mathbf{S}_{\delta_{\mathbf{b}}}, \mathbf{b}_{\text{res}}}^*, & \text{if } r = \frac{d}{2}. \end{cases} \end{aligned}$$

The composition of the two links is exactly $\eta_{\mathbf{S}_{\mathbf{a}}, \mathbf{S}_{\mathbf{b}}}^*$ of the needed indentation degree by (5.15.3) and of the required shift by (5.15.1).

²⁴Here, as before, the shift $\mathbf{t}'_{\mathbf{a}'}$ for a periodic semi-meander \mathbf{a}' for $(\mathbf{S}_{\delta_{\mathbf{b}}}, \mathbf{T}_{\delta_{\mathbf{b}}})$ is taken to be $\mathbf{t}_{\delta_{\mathbf{b}}, \tilde{\mathbf{a}}'}$, where $\tilde{\mathbf{a}}'$ is a periodic semi-meander of (\mathbf{S}, \mathbf{T}) consisting of all the arcs and semi-lines of \mathbf{a}' together with the arc $\delta_{\mathbf{b}}$.

This concludes the proof of Theorem 4.3.

APPENDIX A. COHOMOLOGY OF QUATERNIONIC SHIMURA VARIETIES

We include the proof of Proposition 2.26 regarding the cohomology of our “slightly twisted” quaternionic Shimura varieties. It is based on comparing the cohomology with the known case when $\mathbf{T} = \emptyset$. This is certainly known to the experts, but we could not find the exact version in the literature.

A.1. Discrete Shimura varieties for F^\times . Consider a Deligne homomorphism for $T_{F,\mathbf{T}} := \text{Res}_{F/\mathbb{Q}}(\mathbb{G}_m)$ given by

$$\begin{aligned} h_{\mathbf{T}} : \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times &\longrightarrow T_F(\mathbb{R}) = (\mathbb{R}^\times)^{\mathbf{T}} \times (\mathbb{R}^\times)^{\Sigma_\infty - \mathbf{T}} \\ z &\longmapsto (|z|^2, \dots, |z|^2, (1, \dots, 1)). \end{aligned}$$

Under this choice of Deligne homomorphism, we can define a discrete Shimura variety $\text{Sh}_{K_{T,p}}(T_{F,\mathbf{T}})$ for $K_{T,p} = \mathcal{O}_{\mathfrak{p}}^\times$ whose complex points are given by

$$\text{Sh}_{K_{T,p}}(T_{F,\mathbf{T}})(\mathbb{C}) = F^{\times, \text{cl}} \backslash \mathbb{A}_F^{\infty, \times} / \mathcal{O}_{\mathfrak{p}}^\times.$$

It admits an integral canonical model with special fiber $\text{Sh}_{K_{T,p}}(T_{F,\mathbf{T}})$ over \mathbb{F}_{p^g} (in the sense of [TX13⁺a, Section 2.8]), which is determined by the Shimura reciprocity map

$$\text{Rec}_{T,\mathbf{T},p} : \text{Gal}_{\mathbb{F}_{p^g}} \longrightarrow F^{\times, \text{cl}} \backslash \mathbb{A}_F^{\infty, \times} / \mathcal{O}_{\mathfrak{p}}^\times.$$

Explicitly, $\text{Rec}_{T,\mathbf{T},p}$ sends the geometric Frobenius Frob_{p^g} to the finite idele $(p_F)^{\#\mathbf{T}}$.

Fix a prime number $\ell \neq p$. The algebraic representation $\rho_{T,\mathbf{T}}^w$ of $T_{F,\mathbf{T}} \times \mathbb{C} \cong \prod_{\tau \in \Sigma_\infty} \mathbb{G}_{m,\tau}$ sending x to $(x^{2-w}, \dots, x^{2-w})$ gives a lisse $\overline{\mathbb{Q}}_\ell$ -étale sheaf $\mathcal{L}_{T,\mathbf{T}}^w$ of pure weight $2(w-2)\#\mathbf{T}$ on $\text{Sh}_{K_{T,p}}(T_{F,\mathbf{T}})$.

A.2. Changing \mathbf{T} . We need to compare the Shimura varieties $\text{Sh}_{K_p}(G_{\mathbf{S},\mathbf{T}})$ and $\text{Sh}_{K_p}(G_{\mathbf{S},\emptyset})$. Note that the natural product morphism

$$\text{pr} : G_{\mathbf{S},\emptyset} \times \text{Res}_{F/\mathbb{Q}} \mathbb{G}_m \rightarrow G_{\mathbf{S},\mathbf{T}}$$

is compatible with the Deligne homomorphism $h_{\mathbf{S},\emptyset} \times h_{\mathbf{T}}$ on the source and $h_{\mathbf{S},\mathbf{T}}$ on the target, i.e. $\text{pr} \circ (h_{\mathbf{S},\emptyset} \times h_{\mathbf{T}}) = h_{\mathbf{S},\mathbf{T}}$. This gives a natural morphism of Shimura varieties

$$(A.2.1) \quad \text{pr}_\bullet : \text{Sh}_{K_p}(G_{\mathbf{S},\emptyset}) \times \text{Sh}_{K_{T,p}}(T_{F,\mathbf{T}}) \longrightarrow \text{Sh}_{K_p}(G_{\mathbf{S},\mathbf{T}}).$$

Moreover, the product morphism is compatible with the algebraic representations in the sense that

$$\rho_{\mathbf{S},\mathbf{T}}^{(k,w)} \circ \text{pr} \cong \rho_{\mathbf{S},\emptyset}^{(k,w)} \boxtimes \rho_{T,\mathbf{T}}^w.$$

So we have a natural isomorphism of sheaves

$$(A.2.2) \quad \text{pr}_\bullet^*(\mathcal{L}_{\mathbf{S},\mathbf{T}}^{(k,w)}) \cong \mathcal{L}_{\mathbf{S},\emptyset}^{(k,w)} \boxtimes \mathcal{L}_{T,\mathbf{T}}^w.$$

Proposition A.3. *Let $\pi \in \mathcal{A}_{(k,w)}$ be an automorphic representation appearing in the cohomology of the Shimura variety $\text{Sh}_K(G_{\mathbf{S},\mathbf{T}})$. Then we have a canonical isomorphism*

$$H_{\text{et}}^i(\text{Sh}_K(G_{\mathbf{S},\mathbf{T}})_{\overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{S},\mathbf{T}}^{(k,w)})[\pi] = \begin{cases} \rho_{\pi,\mathfrak{p}}^{\otimes d} \otimes [\det(\rho_{\pi,\mathfrak{p}})(1)]^{\otimes \#\mathbf{T}} & \text{if } i = d, \\ 0 & \text{if } i \neq d; \end{cases}$$

which is, up to semi-simplification, equivariant for the action of the geometric Frobenius Frob_{p^g} .

Proof. The proposition is known when $\mathbf{T} = \emptyset$ by [BL84, §3.2] (note that we have the tensor product instead of tensorial induction because $\rho_{\pi, \mathbf{p}}$ is unramified at \mathbf{p} .) For general \mathbf{T} , the morphism (A.2.1) induces an isomorphism

$$\begin{aligned} H_{\text{et}}^*(\text{Sh}_K(G_{\mathbf{S}, \mathbf{T}})_{\overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{S}, \mathbf{T}}^{(k, w)}) &\cong H_{\text{et}}^*(\text{Sh}_K(G_{\mathbf{S}, \mathbf{T}})_{\overline{\mathbb{F}}_p} \times \text{Sh}_{K_{T, p}}(T_{F, \mathbf{T}})_{\overline{\mathbb{F}}_p}, \text{pr}_*^*(\mathcal{L}_{\mathbf{S}, \mathbf{T}}^{(k, w)}))_{\mathbb{A}_F^{\infty, \times}} \\ &\stackrel{(A.2.2)}{\cong} \left(H_{\text{et}}^*(\text{Sh}_K(G_{\mathbf{S}, \emptyset})_{\overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{S}, \emptyset}^{(k, w)}) \otimes H_{\text{et}}^0(\text{Sh}_{K_{T, p}}(T_{F, \mathbf{T}})_{\overline{\mathbb{F}}_p}, \mathcal{L}_{T, \mathbf{T}}^w) \right)_{\mathbb{A}_F^{\infty, \times}}, \end{aligned}$$

where the superscript $\mathbb{A}_F^{\infty, \times}$ means to take the invariant part for the *anti-diagonal action* of this group, i.e. $z \in \mathbb{A}_F^{\infty, \times}$ acts by (z, z^{-1}) . So, if ω_π denotes the central character of π , we have

$$H_{\text{et}}^*(\text{Sh}_K(G_{\mathbf{S}, \mathbf{T}})_{\overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{S}, \mathbf{T}}^{(k, w)})[\pi] \cong H_{\text{et}}^*(\text{Sh}_K(G_{\mathbf{S}, \emptyset})_{\overline{\mathbb{F}}_p}, \mathcal{L}_{\mathbf{S}, \emptyset}^{(k, w)})[\pi] \otimes H_{\text{et}}^0(\text{Sh}_{K_{T, p}}(T_{F, \mathbf{T}})_{\overline{\mathbb{F}}_p}, \mathcal{L}_{T, \mathbf{T}}^w)[\omega_\pi],$$

where the last factor is the one-dimensional subspace where $\mathbb{A}_F^{\infty, \times}$ acts through ω_π . By the Shimura reciprocity map $\text{Rec}_{T, \mathbf{T}, p}$ recalled in Subsection A.1 and the Eichler–Shimura relation (2.5.1), the geometric Frobenius Frob_{p^g} acts on this one-dimensional space by multiplication by

$$\omega_\pi(\underline{p}_F)^{-\#\mathbf{T}} = (\det(\rho_{\pi, \mathbf{p}}(\text{Frob}_{p^g}))/p^g)^{\#\mathbf{T}}.$$

This concludes the proof of this Proposition. \square

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